1. (5 pts) Recalling that \( V = V_+ + V_- \), \( I = I_+ + I_- \), \( Z_o = \frac{V_+}{I_+} = -\frac{V_-}{I_-} \), and \( Y_o = 1/Z_o \), show that:

\[
V_+ = \frac{V + I \cdot Z_o}{2} \quad V_- = \frac{V - I \cdot Z_o}{2} \quad I_+ = \frac{I + V \cdot Y_o}{2} \quad I_- = \frac{I - V \cdot Y_o}{2}
\]

\( V_+ = \frac{V + I \cdot Z_o}{2} : \)

\[
I = I_+ + I_- = \frac{V_+}{Z_o} + \left( -\frac{V_-}{Z_o} \right) \Rightarrow V_- = Z_o \cdot \left( \frac{V_+}{Z_o} - I \right) = V_+ - Z_o \cdot I
\]

\[\Rightarrow V = V_+ + V_- = V_+ + (V_+ - Z_o \cdot I) \Rightarrow V_+ = \frac{V + I \cdot Z_o}{2}\]

\( V_- = \frac{V - I \cdot Z_o}{2} : \)

\[
I = I_+ + I_- = \frac{V_+}{Z_o} + \left( -\frac{V_-}{Z_o} \right) \Rightarrow V_+ = Z_o \cdot \left( \frac{V_-}{Z_o} + I \right) = V_+ + Z_o \cdot I
\]

\[\Rightarrow V = V_+ + V_- = (V_+ + Z_o \cdot I) + V_- \Rightarrow V_- = \frac{V - I \cdot Z_o}{2}\]

\( I_+ = \frac{I + V \cdot Y_o}{2} : \)

\[
V = V_+ + V_- = Z_o \cdot I_+ + (-Z_o \cdot I_+) = \frac{I_+}{Y_o} + \left( -\frac{I_-}{Y_o} \right) \Rightarrow I_- = Y_o \cdot \left( \frac{I_+}{Y_o} - V \right) = I_+ - Y_o \cdot V
\]

\[\Rightarrow I = I_+ + I_- = I_+ + (I_+ - Y_o \cdot V) \Rightarrow I_+ = \frac{I + V \cdot Y_o}{2}\]

\( I_- = \frac{I - V \cdot Y_o}{2} : \)

\[
V = V_+ + V_- = Z_o \cdot I_+ + (-Z_o \cdot I_+) = \frac{I_+}{Y_o} + \left( -\frac{I_-}{Y_o} \right) \Rightarrow I_+ = Y_o \cdot \left( \frac{I_-}{Y_o} + V \right) = I_- + Y_o \cdot V
\]

\[\Rightarrow I = I_+ + I_- = (I_+ + Y_o \cdot V) + I_- \Rightarrow I_- = \frac{I - V \cdot Y_o}{2}\]
2. (10 pts) A pulse generator with internal impedance 50Ω is connected to the transmission line arrangement shown below. When the generator is turned on it begins sending voltage pulses down the line, one pulse every 20nsec.

Use the following notation:

\( \rho \) = reflection coefficient for waves incident from a 50 Ω section to a 150 Ω section
\( \tau \) = transmission coefficient from a 50 Ω section to a 150 Ω one
\( \rho' \) = reflection coefficient from a 150 Ω section to a 50 Ω one
\( \tau' \) = transmission coefficient from a 150 Ω section to a 50 Ω one
\( V_{+i} \) = forward pulse voltage amplitude in section i
\( V_{-i} \) = backward pulse voltage amplitude in section i

At \( t = 0 \) the first pulse sent from the generator is at interface A.
If the pulses sent from the generator have height \( V_{+1} \), what is \( V_{+2} \) at:
- \( t = 5 \) nsec
- \( t = 25 \) nsec
- \( t = 35 \) nsec
- \( t = 45 \) nsec
in steady state (i.e. \( t = \infty \))?
In steady state, what are \( V_{+3} \) and \( V_{-1} \)?
Solution:
First note that any fraction of a pulse that bounces back to the generator won’t bounce back since it will be “absorbed” by the matched load at the generator. Also, since section three stretches to infinity, we don’t have to worry about any pulse bouncing back from the “far end” (i.e., $V_3 = 0$).

In picture form, here’s what’s going on:
In table form, the summary is:

<table>
<thead>
<tr>
<th>time (nsec)</th>
<th>V -1</th>
<th>V +2</th>
<th>V -2</th>
<th>V +3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>ρ·V 1</td>
<td>τ·V 1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>0</td>
<td>ρ'·(V +2 from the first pulse) = ρ'·(τ·V +1)</td>
<td>0</td>
<td>τ'·(V +2 from the first pulse) = τ'·(τ·V +1)</td>
</tr>
<tr>
<td>25</td>
<td>τ'·(V -2 from the first pulse) + ρ·(V +1 from the second pulse out of the generator) = ρ·V +1 + τ·(ρ·(τ·V +1))</td>
<td>0</td>
<td>τ'·(V -2 from the last series of pulses) = τ'·(τ·V +1) + τ·V +1</td>
<td></td>
</tr>
<tr>
<td>35</td>
<td>0</td>
<td>ρ'·(V -2 from the last series of pulses) + τ·(V +1 from the third pulse out of the generator) = (ρ'·ρ·τ·V +1) + τ·V +1</td>
<td>0</td>
<td>τ'·(V +2 from the last series of pulses) = τ'·(τ·V +1) + τ·V +1</td>
</tr>
<tr>
<td>45</td>
<td>45</td>
<td>ρ'·(V -2 from the last series of pulses) + τ·(V +1 from the third pulse out of the generator) = (ρ'·ρ·τ·V +1) + τ·V +1</td>
<td>0</td>
<td>τ'·(V +2 from the last series of pulses) = τ'·(τ·V +1) + τ·V +1</td>
</tr>
</tbody>
</table>

Following the pattern we have for V +2

\[
\begin{align*}
\text{t = 5 nsec} & \quad V_{+2} = \tau \cdot V_{+1} \\
\text{t = 25 nsec} & \quad V_{+2} = \rho' \cdot \rho' \cdot \tau \cdot V_{+1} + \tau \cdot V_{+1} = \tau \cdot V_{+1} \cdot \left[ 1 + (\rho')^2 \right] \\
\text{t = 45 nsec} & \quad V_{+2} = \rho' \cdot \left[ \rho' \cdot \rho' \cdot \tau \cdot V_{+1} + \tau \cdot V_{+1} \right] + \tau \cdot V_{+1} = \tau \cdot V_{+1} \cdot \left[ 1 + (\rho')^2 + (\rho')^4 \right] \\
\text{t = 65 nsec} & \quad V_{+2} = \tau \cdot V_{+1} \cdot \left[ 1 + (\rho')^2 + (\rho')^4 + (\rho')^6 \right]
\end{align*}
\]
Now recalling the Taylor series expansion:

\[
\frac{1}{1 - x^2} = 1 + x^2 + x^4 + x^6 + x^8 + \cdots \quad \text{if } |x| < 1
\]

We see that if we wait for an infinite number of pulses (i.e., in “steady state”) we’ll get:

\[
V_{+2} = \tau \cdot V_{+1} \frac{1}{1 - (\rho)^2}
\]

Here we have \(\rho' = \frac{50 - 150}{50 + 150} = -\frac{1}{2}\) and \(\tau = 1 + \rho = 1 + \frac{150 - 50}{150 + 50} = \frac{3}{2}\) so finally we get

\[
V_{+2} = \tau \cdot V_{+1} \frac{1}{1 - (\rho')^2} = \frac{3}{2} \cdot \frac{1}{1 - \frac{1}{4}} \cdot V_{+1} = 2 \cdot V_{+1}
\]

and then we can also get:

\[
V_{+3} = \tau' \cdot V_{+2} = \frac{1}{2} \cdot (2 \cdot V_{+1}) = V_{+1}
\]

\[
V_{-2} = \rho' \cdot V_{+2} = -\frac{1}{2} \cdot (2 \cdot V_{+1}) = -V_{+1}
\]

\[
V_{-1} = \rho \cdot V_{+1} + \tau' \cdot V_{-2} = \frac{1}{2} \cdot V_{+1} + \frac{1}{2} \cdot (-V_{+1}) = 0
\]

So in steady state the “whole” pulse gets through, with no net reflection back to the generator!
3. (5 pts) To find the phasor form for the voltage and current along a transmission line (i.e., the Telegraphist’s equations in “time harmonic form”) you can replace \( \frac{\partial}{\partial t} \) by \( j\omega \) in:

\[
\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t} \quad \frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}
\]

Make this substitution and show that

\[
V = V_+ e^{-j\beta z} + V_- e^{j\beta z} \quad I = \frac{1}{Z_o} \left[ V_+ e^{-j\beta z} - V_- e^{j\beta z} \right]
\]

are solutions to the Telegraphist’s equations so long as \( \beta = \omega \sqrt{LC} \).

[Problem taken from Ramo, problem 5.7a (5.5a in 2nd ed.)]

Solution:

After assuming all time dependence is \( e^{j\omega t} \), the Telegraphist’s equations become:

\[
\frac{\partial V}{\partial z} = -L(j\omega)I \quad \frac{\partial I}{\partial z} = -C(j\omega)V
\]

Let’s substitute the proposed solutions

\[
V = V_+ e^{-j\beta z} + V_- e^{j\beta z} \quad I = \frac{1}{Z_o} \left[ V_+ e^{-j\beta z} - V_- e^{j\beta z} \right]
\]

into eq. (1). First we need the \( z \) partial derivative:

\[
\frac{\partial}{\partial z} V = (-j\beta) V_+ e^{-j\beta z} + (j\beta) V_- e^{j\beta z}
\]

Substitution into (1) gives:

\[
(-j\beta) V_+ e^{-j\beta z} + (j\beta) V_- e^{j\beta z} = -L(j\omega)I = -L(j\omega) \left[ \frac{1}{Z_o} \left( V_+ e^{-j\beta z} - V_- e^{j\beta z} \right) \right]
\]

The \( = \) symbol indicates we need to check and see if the equality actually holds.

Recall \( Z_o = \sqrt{\frac{L}{C}} \) so

\[
(-j\beta) V_+ e^{-j\beta z} + (j\beta) V_- e^{j\beta z} = -L(j\omega) \left[ \sqrt{\frac{C}{L}} \left( V_+ e^{-j\beta z} - V_- e^{j\beta z} \right) \right]
\]

Multiplying out the right hand side gives

\[
(-j\beta) V_+ e^{-j\beta z} + (j\beta) V_- e^{j\beta z} = -\sqrt{LC} (j\omega) V_+ e^{-j\beta z} - \left( -\sqrt{LC} (j\omega) \right) V_- e^{j\beta z}
\]

So for the equality to hold we need:

\[
(-j\beta) = -\sqrt{LC} (j\omega) \Rightarrow \beta = \omega \sqrt{LC} \quad \text{and} \quad j\beta = \sqrt{LC} (j\omega) \Rightarrow \beta = \omega \sqrt{LC}
\]

You can go through a similar substitution for eq. (2), reaching the same conclusion.
4. (5 pts) For a transmission line “terminated” at \( z = 0 \) by a load impedance \( Z_L \) and recalling:

\[
Z_o = \frac{V_+}{I_+} = -\frac{V_-}{I_-}, \quad \rho(z=-l) = \frac{V_- \cdot \exp(-j\beta l)}{V_+ \cdot \exp(j\beta l)}, \quad Z(z=-l) = \frac{V_- \cdot \exp(j\beta l) + V_+ \cdot \exp(-j\beta l)}{I_- \cdot \exp(j\beta l) + I_+ \cdot \exp(-j\beta l)}
\]

show that

\[
Z(z=-1) = Z_o \frac{1 + \rho(z=0) \cdot \exp(-j2\beta l)}{1 - \rho(z=0) \cdot \exp(-j2\beta l)} = Z_o \frac{Z_L + Z_o \cdot \tanh(j\beta l)}{Z_o + Z_L \cdot \tanh(j\beta l)} = Z_o \frac{Z_L + jZ_o \cdot \tan(\beta l)}{Z_o + jZ_L \cdot \tan(\beta l)}
\]

Solution:

Start with

\[
Z(z=-l) = \frac{V}{I} = \frac{V_+ \cdot e^{j\beta l} + V_- \cdot e^{-j\beta l}}{I_+ \cdot e^{j\beta l} + I_- \cdot e^{-j\beta l}} = \frac{V_+ \cdot e^{j\beta l} \left(1 + \frac{V_- \cdot e^{-j\beta l}}{V_+ \cdot e^{j\beta l}}\right)}{I_+ \cdot e^{j\beta l} \left(1 + \frac{I_- \cdot e^{-j\beta l}}{I_+ \cdot e^{j\beta l}}\right)} = \frac{V_+ \left(1 + \frac{V_-}{V_+} \cdot e^{-j2\beta l}\right)}{I_+ \left(1 + \frac{I_-}{I_+} \cdot e^{-j2\beta l}\right)}
\]

Now recall that \( \frac{V_-}{V_+} = \rho(0) = -\frac{I_-}{I_+} \) and \( \frac{V_+}{I_+} = Z_o \), we have:

\[
Z(z=-l) = Z_o \frac{1 + \rho(0) \cdot e^{-j2\beta l}}{1 - \rho(0) \cdot e^{-j2\beta l}}
\]

Now recall that \( \rho(0) = \frac{Z_L - Z_o}{Z_L + Z_o} \), then substituting into the expression above:

\[
Z(z=-l) = Z_o \frac{1 + \rho(0) \cdot e^{-j2\beta l}}{1 - \rho(0) \cdot e^{-j2\beta l}} = Z_o \frac{1 + \frac{Z_L - Z_o}{Z_L + Z_o} \cdot e^{-j2\beta l}}{1 - \frac{Z_L - Z_o}{Z_L + Z_o} \cdot e^{-j2\beta l}}
\]

\[
= Z_o \frac{Z_L + Z_o + Z_L \cdot e^{-j2\beta l} - Z_o \cdot e^{-j2\beta l}}{Z_L + Z_o - Z_L \cdot e^{-j2\beta l} + Z_o \cdot e^{-j2\beta l}}
\]

\[
= Z_o \frac{Z_L \cdot \left(e^{j\beta l} + e^{-j\beta l}\right) + Z_o \cdot \left(e^{j\beta l} - e^{-j\beta l}\right)}{Z_o \cdot \left(e^{j\beta l} + e^{-j\beta l}\right) + Z_L \cdot \left(e^{j\beta l} - e^{-j\beta l}\right)}
\]

\[
= Z_o \frac{Z_L + Z_o \cdot \tanh(\beta \cdot l)}{Z_o + Z_L \cdot \tanh(\beta \cdot l)}
\]