Chapter 4

Conformal Mapping

Two dimensional electro-magnetic field problems constitute a major part of transmission line analysis. Complex function theory has been used widely to solve two dimensional fields, in the sense that complex functions can represent two dimensional field components in the complex plane. For example if the real part of a function gives the electric fields in two dimensions then the imaginary part gives the magnetic fields. It turns out that complex functions suitable for representing fields satisfy Laplace's equation, and real and imaginary parts of the functions are orthogonal to each other, thus if one part represents the field lines, the other part represents equipotential lines. Another very useful property of complex functions is that points in one complex plane can be mapped to another complex plane by means of a complex function. Using this property transmission line traces of regular shapes can be mapped into parallel plates, with all the surrounding space in the original domain in between the plates in the mapped domain. This kind of mapping is called a conformal map, since it preserves local shape and angle. The transmission line parameters like capacitance and inductance calculated from the geometries are the same for both domains. For some cases where a suitable map exists, conformal mapping proves to be a very efficient method in calculating transmission line parameters. In the mapped domain the solution to Laplace's equation is straight forward if the map generates parallel plates with all the surrounding space in the real domain mapped in between.

Conformal mapping is a very well established method used to solve two dimensional field problems with perfect conductors present. In practice, conformal mapping can be used to calculate high frequency parameters of transmission lines[7]. In this chapter, applications of conformal mapping to lossy transmission lines with real conductors will be presented. With this method dc to high frequency resistance has been obtained with good agreement to more rigorous solutions and experimental data.

4.1 Properties of Conformal Mapping

A conformal map is an analytical function from one complex plane to another complex plane. The complex plane which is the domain of the map is usually called the z-plane and the range of the map is called the w-plane. If a complex function is continuous and has a unique derivative then this function is a map. Let w be a map from z-plane to w-plane, the derivative dw/dz yields the Cauchy-Riemann conditions,

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y}, \end{aligned} \tag{4.1}$$

where $w = u + jv$ and $z = x + jy$. For any complex function (4.1) is the necessary and sufficient condition for being a conformal map.

As an example let us consider a coaxial cable with inner radius b and

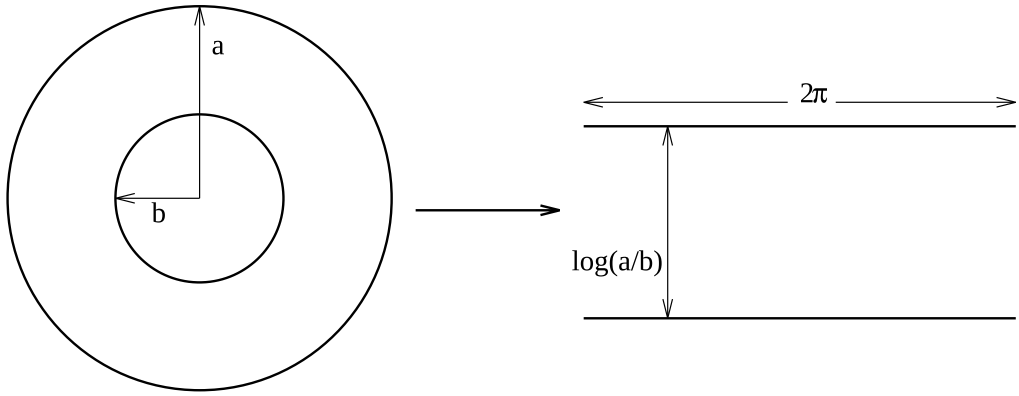

Figure 4.1: Mapping of coaxial cable to parallel plates, the map is $w = \log z$

outer radius a. The cable can be mapped into parallel plates with the map,

$$w = \log z \tag{4.2}$$

Each conductor contour maps into a straight plate, space between conductors maps between the parallel plates. In the mapped domain, the inductance and capacitance can be calculated easily. From the map, the separation between parallel plates is $\log(a/b)$. The plates are 2π wide. Characteristic impedance of this structure can be calculated as

$$\begin{aligned} C &= \epsilon_0 \epsilon_r \frac{2\pi}{\log a/b} \\ L &= \mu_0 \frac{\log a/b}{2\pi} \\ Z &= \sqrt{\frac{L}{C}} = \frac{60}{\sqrt{\epsilon_r}} \log a/b. \end{aligned} \tag{4.3}$$

For a lossy conductor, mapping function should be applied not only to conductor traces but also to the inside of conductors. At dc and low frequencies, the currents will be flowing over the whole cross-section, whereas for a perfect conductor current is flowing on the surface. Mapping boundaries of conductors is enough for the perfect conductor case. For conductors with finite conductivity, use of mapping is limited to high frequency since only then the current is flowing almost at the surface. A method of using conformal map with finite conductivity at lower frequencies is presented in the following sections.

4.2 Scaled Conductivity

Application of the conformal mapping technique to find conductor losses has been used before in reference [7] by using the mapping function to solve for current distribution at high frequency. The current distribution was used in conjunction with skin-depth limited surface impedance to calculate loss. The assumption made here is that the frequency is high enough that cross sectional dimensions of conductors are much larger than the skin-depth. Since this technique is a high frequency technique, transition from dc to high frequency resistance can not be obtained. To extend lossy transmission line analysis using conformal mapping to low frequencies, all conductor cross sections should be mapped.

Conformal mapping preserves angles and local shape so, an infinitesimal rectangle with sides Δx and Δy will map to a rectangle Δw_x andΔw_y. Conductance of this infinitesimal rectangle will be

$$\begin{aligned} g_{org} &= \sigma \Delta x \Delta y \\ g_{map} &= \sigma_m \Delta w_x \Delta w_y, \end{aligned} \tag{4.4}$$

where g_{org} and g_{map} are conductances in the original and mapped domain respectively, and σ_m is the conductivity in the map domain. Since both rectangles in the original and mapped domains represent the same amount of conductor, conductance of both rectangles should be the same (i.e., $g_{org} = g_{map}$).

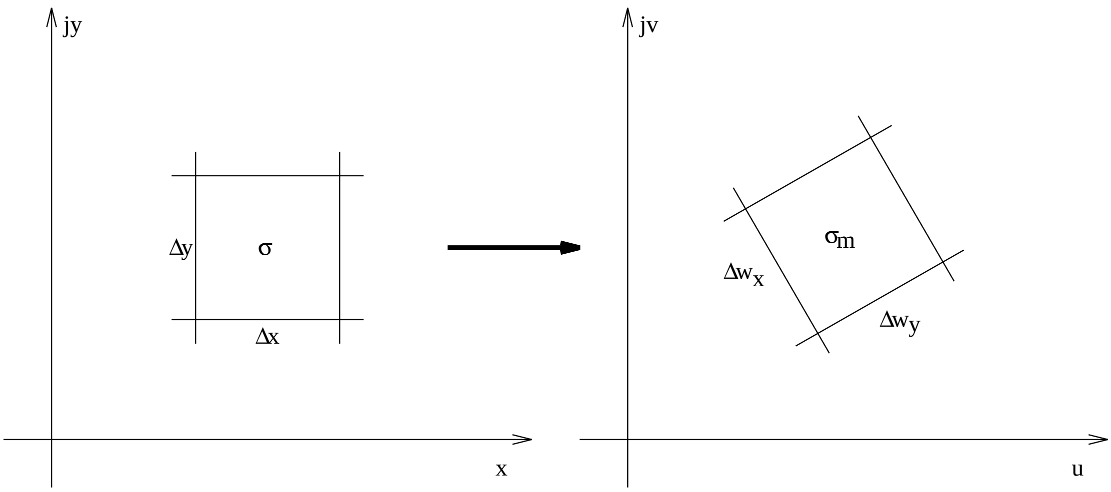

Figure 4.2: Mapping an infinitesimal rectangle into another rectangle, map introduces rotation and scaling

The relation between the original and mapped domain quantities can be derived from Cauchy-Riemann conditions given by (4.1). The infinitesimal line segment that corresponds to Δx is

$$\Delta w_x = \Delta x \frac{\partial v}{\partial y} - j \Delta x \frac{\partial u}{\partial y}. \tag{4.5}$$

A similar expression can be obtained for the infinitesimal line segment that corresponds to Δy;

$$\Delta w_y = -\Delta y \frac{\partial v}{\partial x} + j \Delta y \frac{\partial u}{\partial x}. \tag{4.6}$$

It can be verified that segments Δw_x and Δw_y are orthogonal. The map introduces a rotation and a scale factor. The lengths of line segments are found to be

$$\begin{aligned} |\Delta w_x| &= \Delta x \left| \frac{\partial (v - ju)}{\partial y} \right| \\ |\Delta w_y| &= \Delta y \left| \frac{\partial (-v + ju)}{\partial x} \right|. \end{aligned} \tag{4.7}$$

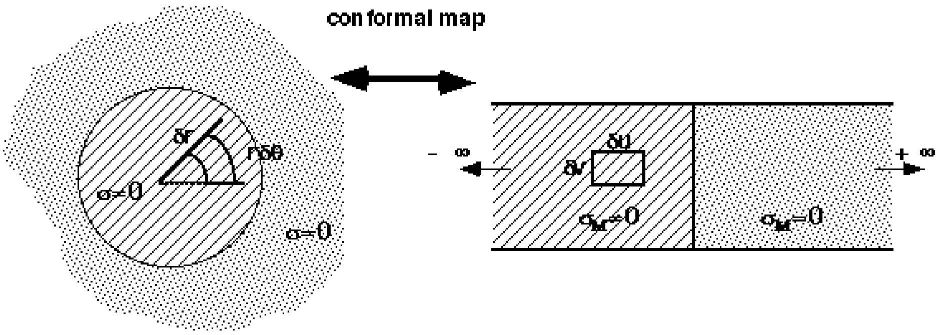

Figure 4.3: Conformal mapping of a circular wire. Inside of the conductor maps to semi-infinite strip from zero to negative infinity.

The partial derivatives in the above equations can be identified as the total derivative $|dw/dz|$ since the map is differentiable and value of derivative is independent of direction. Substituting (4.7) in (4.4), the conductivity in the mapped domain must be,

$$\sigma_m = \frac{\sigma}{\left|\frac{dw}{dz}\right|^2}. \tag{4.8}$$

A scaling factor M can be defined as $|dw/dz|$. Equation (4.8) can be rewritten as $\sigma_m = \sigma/M^2$. In general M is a function of both x and y (or u and v).

4.2.1 Example: Single cylindrical conductor

As an example, consider a circular wire of radius r_0 and conductivity σ. At dc, the resistance of the wire is $1/(\pi\sigma r_0^2)$. The wire can be mapped into a semi-infinite strip with the map (Fig. 4.3),

$$w = \log\frac{z}{r_0} = \log\left(\frac{r}{r_0}\right) + j\theta, \tag{4.9}$$

where the right most equation is in polar coordinates in the original domain. With the map given by (4.9), the inside of the wire would map into a band, 2π wide and extending from 0 to $-\infty$. Outside of the wire similarly maps into a band from 0 to $+\infty$. The surface of the conductor maps to $u = 0$, from $-\pi$ to π on the v-axis. The center of the conductor maps at negative infinity. Application of (4.8) defines σ_m for the conductor region in the mapped domain,

$$\sigma_m = \sigma r^2 = \sigma r_0^2 e^{2u}. \tag{4.10}$$

The conductivity in the mapped domain σ_m in a way represents the area scaling of the map. In the mapped domain, dc resistance is calculated using the integral,

$$R_{map_{dc}} = \left(\int_0^{-\infty} \int_{-\pi}^{\pi} \sigma_m du dv \right)^{-1}. \tag{4.11}$$

Evaluating the integral, $R_{map_{dc}}$ is found to be equal to the dc resistance calculated in the original domain.

Using the conformal mapping technique in practical applications to calculate the dc resistance is not attractive. Mapping regions is not easy for common structures like rectangles. One way of overcoming these difficulties is to use an effective internal impedance introduced in the previous chapter at the surface that would represent the inside of conductors.

4.3 Transverse Resonance Method in Mapped Domain

To find the surface impedance of the conformally mapped conductor, a nonuniform transmission line problem must be solved. For the circular wire example, applying transverse resonance, the surface impedance is just the input impedance of an equivalent transmission line looking from $u = 0$ in the $-u$ direction, with uniform plate separation of one unit, 2π wide, filled with a conducting medium with conductivity given by (4.10). Note that σ_m is independent of v for this case, hence the transverse resonance technique can be applied. This equivalent transmission line extends to negative infinity. The propagation constant varies with position u along the line, and is

$$\gamma(u) = \sqrt{j\omega\mu\sigma_m} \tag{4.12}$$

where ω is the angular frequency, μ is the permeability of the conductor, and σ_m is given by (4.10). The characteristic impedance at each position u is

$$Z_0(u) = \frac{1}{2\pi}\sqrt{\frac{j\omega\mu}{\sigma_m}}. \tag{4.13}$$

Solution for the input impedance can be found by setting the boundary condition, I = 1A at $u = 0$, and then solving for voltage using the telegraphers' equations given by (2.2). The input impedance is found to be

$$Z_S^{circ} = j\sqrt{\frac{j\omega\mu}{\sigma}}\frac{\mathbf{J}_0(jr_0\sqrt{j\omega\mu\sigma})}{\mathbf{J}_1(jr_0\sqrt{j\omega\mu\sigma})} \tag{4.14}$$

where $\mathbf{J}_0$ and $\mathbf{J}_1$ are the 0^{th} and 1^{st} order Bessel functions of the first kind, respectively. This surface impedance found in the conformally mapped plane is identical to the exact expression found by solving directly the Helmholtz equation in the cylindrical coordinates for the original, cylindrically-symmetric conductor [28].

4.4 Conductor Loss Calculations Using Conformal Mapping

In general, the nonuniform conducting medium problem resulting from a conformal map is more difficult than the original problem. Assuming that the frequency dependent current distribution in the conductors are governed by the sum of two effects, a some what simpler approach is possible for conductor loss calculations. The first one is frequency dependent skin-depth, which is determined by the conductor geometry, conductivity, permeability and the excitation

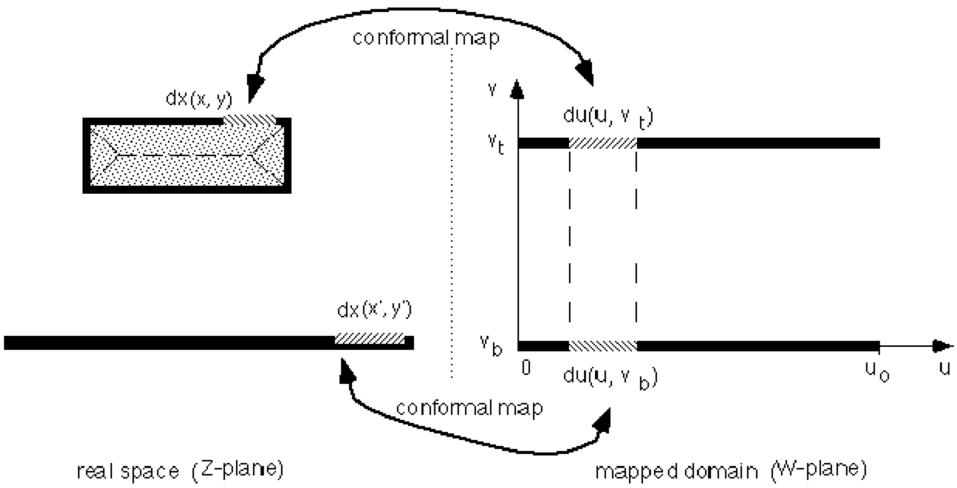

Figure 4.4: Conformal mapping process which unfolds conductors in the real space (z-plane), scaling conductor conductivity and effective internal impedance Z_{eii} into the mapped plane (w-plane)

frequency, which leads to a complex effective internal impedance function on surface of the conductor. The second effect is the current crowding induced by the interaction between the conductors (i.e., the proximity effect). To accurately predict the behavior of the transmission line, the model should account for uniform current distribution at low frequency and at high frequency it should account for redistribution of the current due to skin and proximity effects.

To use conformal mapping effectively in the scheme explained above, conductors in the real-space domain can be replaced by impedance shells delineating the boundaries. These shells then can be mapped into parallel plates using conformal mapping. Skin-depth effects would be accounted for in the real-space domain and the current crowding effects in the mapped domain.

In general, assume a conformal map $f(z)$ has been found for a particular transmission line structure. The map is such that it produces parallel plates in the w-plane. Referring to Fig. 4.4, consider a point (x, jy) on the surface of the real space conductor with corresponding effective internal impedance $Z_{eii}(x, y)$. Assuming this point maps onto a point in the w-plane at (u, jv_t), on the top plate, the scaled effective internal impedance in the w-plane is given by,

$$M(u, v_t)Z_{eii}(x, y) = M(u, v_t)Z_{eii}(\Re\{f^{-1}(u, v_t)\}, \Im\{f^{-1}(u, v_t)\}) \qquad (4.15)$$

where $f^{-1}(u, v)$ is the inverse of the mapping function, $\Re$ and $\Im$ return real and imaginary parts of the argument, respectively, and v_t denotes the v-coordinate of the top plate. The differential series impedance per unit length dZ_{top} due to a differential width du of the top plate is then

$$dZ_{top} = \frac{M(u, v_t)Z_{eii}}{du}, \qquad (4.16)$$

where Z_{eii} is the effective internal impedance at the coordinates in the original domain corresponding to the coordinates (u, v_t) in mapped domain. The bottom plate also contributes to the series impedance

$$dZ_{bot} = \frac{M(u, v_b)Z_{eii}}{du}, \qquad (4.17)$$

where v_b denotes the v-coordinate of the bottom plate. Now the transverse resonance technique can be applied for an infinitesimal width on the parallel plates. Assuming uniform magnetic fields between the plates, the inductance linking the top and bottom plate is given by

$$dL = \frac{\mu_0(v_t - v_b)}{du}, \qquad (4.18)$$

where μ_0 is the permeability of free space. Finally, the total differential series impedance per unit length is

$$dZ_{tot} = dZ_{top} + dZ_{bot} + j\omega dL. \qquad (4.19)$$

The total series impedance per unit length $Z(\omega)$ for the transmission line can be calculated by the parallel combination of each differential impedance

$$Z(\omega)=\left[\int_0^{u_0}\frac{du}{j\omega\mu(v_t-v_b)+Z_{eii}(u,v_t)M(u,v_t)+Z_{eii}(u,v_b)M(u,v_b)}\right]^{-1}, \tag{4.20}$$

where u_0 is the plate width in the mapped domain. Previous attempts to use conformal mapping to evaluate conductor loss have used average values for the scale factor [33, 34] and generally have not placed M and the effective internal impedance within the integral for the series impedance. This can lead to significant inaccuracy especially in closely spaced rectangular conductors, where effective internal impedance is a function of position, and proximity effects are dominant.

4.4.1 High Frequency Limit

As a test case, let us investigate (4.20) at high frequency. Except at sharp corners, the impedance on conductor surfaces tends to ([28])

$$Z_{eii}^{hf}=\sqrt{\frac{j\omega\mu}{\sigma}}=\frac{1+j}{\sigma\delta}, \tag{4.21}$$

where δ is the conductor skin depth at radial frequency ω. With this substitution, the high frequency series impedance per unit length for the transmission line is

$$Z^{hf}=\frac{1+j}{\sigma\delta u_0^2}\left(\int_0^{u_0}\left(M(u,v_t)+M(u,v_b)\right)du\right)+j\omega\frac{\mu_0(v_t-v_b)}{u_0}. \tag{4.22}$$

The second term in the sum is identified as being inductive. The inductance is exactly the external inductance,

$$L_{ext}=\frac{\mu_0(v_t-v_b)}{u_0}. \tag{4.23}$$

The first term in (4.22) gives high frequency resistance and a correction to the external inductance due to the finite thickness of the current that is flowing on the surface. At high frequency, resistance can be found if the current distribution is known [7],

$$\frac{I_{tot}^2}{2}\Re\{Z^{hf}\} = \frac{1}{2}\left(\frac{1}{\sigma\delta}\int |J_z|^2 dl\right), \tag{4.24}$$

where I_{tot} is the total current, and the integral is along the surfaces of the conductors (in the z-plane). In the mapped domain, under scaling, dl becomes $du/M(w)$, and $|J_z|$ is given in reference [7]

$$|J_z| = \frac{I_{tot}}{u_0}\left|\frac{dw}{dz}\right| = \frac{I_{tot}M}{u_0}. \tag{4.25}$$

Substituting (4.25) in (4.24), high frequency resistance can be written as

$$\Re\{Z^{hf}\} = \frac{1}{\sigma\delta u_0^2}\int_0^{u_0}\left(M(u,v_t) + M(u,v_b)\right)du. \tag{4.26}$$

which is identical to the real part of (4.22). In order to verify the proposed method, the results are compared to experiment for different geometries in the next section.

4.5 Examples

In the previous sections it has been shown that conformal mapping can be used to calculate the series impedance of a transmission line with lossy conductors at any frequency starting at dc. In this section, the method is tested against a number of geometries and the results are compared to measurements. The transmission lines were fabricated from 12 gauge copper circular wires, co-planar strip lines on a printed circuit board, and hollow square brass tubing.

All were measured with an HP4194A Impedance/Gain-Phase analyzer. This set of structures span a wide range of practical applications. The circular structure could be an example of twisted pair interconnect between circuit boards, coplanar strip lines can represent interconnect traces on printed circuit boards and multi-chip modules, and square bars can be thought as thick traces of conductors in on-chip interconnects.

4.5.1 Experiment

The first transmission line measured was formed from 12 gauge copper wire as follows: two pieces of wire were taped together using electrical insulator tape. The length of the transmission line was 183cm. At one end, the wires are shorted by stripping the insulator and soldering two tips together. At the other end, the insulator was scrapped of for about an inch and bent in a right angle to form a 'T' for the text fixture on the Impedance/Gain-Phase analyzer. The cross-section and dimensions are shown in Fig. 4.5.

The coplanar strip lines were fabricated on a printed circuit board with half ounce copper (which corresponds to 17μm copper thickness) using scotch tape as the mask and copper etchant. After fabricating traces, one end was shorted. Two pieces of 12 gauge copper wire about an inch long were soldered at the other end to each of the traces to attach to the text fixture. The dimensions are in Fig. 4.5.

As for the last case, square brass tubing supplied from a hobby shop was used. The brass tubes were attached together to form a transmission line by scotch tape using microscope slides as spacers. One end was shorted and 12 gauge wire was used as terminals on the other end. The dimensions are also in

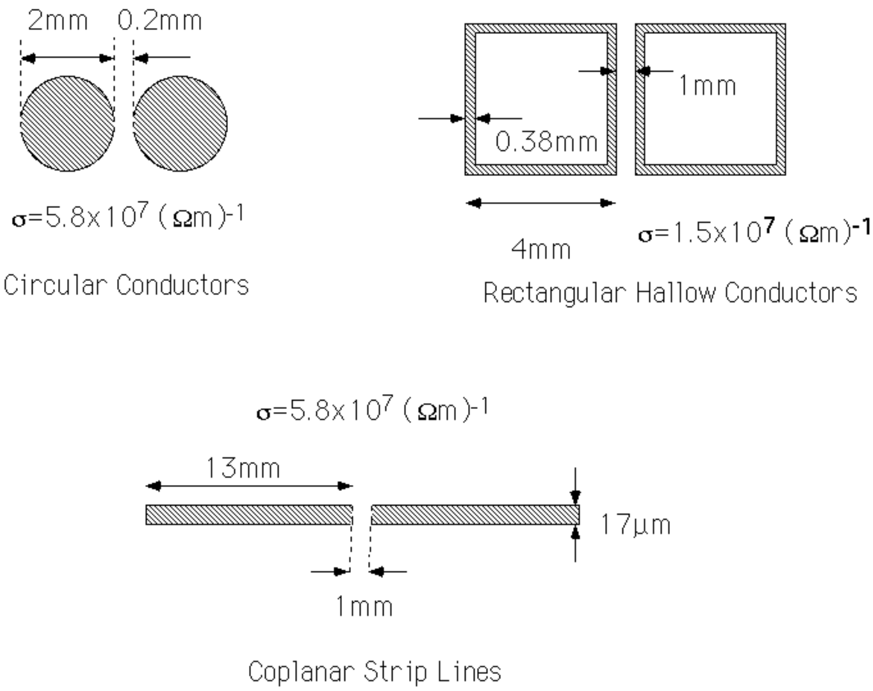

Figure 4.5: Cross-sections of structures used in the experiment, conductivities of metals are given next to cross sections.

Fig. 4.5.

The measurements were made using an Hewlett-Packard Model 4194A Impedance/Gain-Phase Analyzer [16]. The 4194A can measure complex impedance and admittance from 100Hz to 40MHz. The test structures were connected to the Impedance/Gain-Phase analyzer via the text fixture supplied as an accessory (HP model number 16047D). The calibration was done by measuring the impedance of a 1.5in long 12 gauge wire which represents the terminals of the transmission lines, and subtracting this measurement from the transmission line measurements. The measurement of the short also includes contributions from the internal parts and the test fixture itself.

4.5.2 Circular Cylindrical Conductors

The characteristics of parallel wires, or "twin lead," can be calculated by conformal mapping. At high frequencies, apart from the skin-effect which pushes the current to the surface, there is also the proximity effect which pulls the current towards the inner sides of conductors. At low frequency, the current flows uniformly throughout the cross-section of conductors. The effective internal impedance of each circular conductor is given by (4.14) which has the desired low and high frequency behavior. Assuming conductors with radius R and separation $2d$, the desired mapping is

$$w = \pi + j \ln \left(\frac{z-a}{z+a} \right), \tag{4.27}$$

where $a = \sqrt{d^2 - R^2}$. The above map results in a pair of parallel plates, 2π wide in the u-direction and $\ln(d+a)/R$ apart. The scaling factor for this map along the top and bottom plates is

$$M(u) = \frac{1}{a} \left(\frac{d}{R} + \cos(u) \right). \tag{4.28}$$

Substituting in (4.20), the total impedance per unit length can be calculated,

$$Z(\omega) = 2 \left[\int_0^{2\pi} \frac{du}{j\omega\mu \ln\left(\frac{d+a}{R}\right) + Z_{eii} M(u)} \right]^{-1}. \tag{4.29}$$

The above equation is valid throughout all frequency range from dc to the quasi-TEM limit. At high frequencies, (4.29) reduces to

$$Z^{hf} = \frac{\sqrt{j\omega\mu}}{\sqrt{\sigma\pi} R} \frac{d}{a} + j\omega \frac{\mu \ln\left((d+a)/R\right)}{\pi}, \tag{4.30}$$

which is the expected result [28]. The real part of high frequency impedance is increased by a factor of d/a from the simple skin-effect resistance. This increase accounts for current crowding. The second term in (4.30) is the inductive term and the inductance is equal to the external inductance of the structure. The low frequency limit of (4.29) is

$$Z^{lf} = \frac{2}{\sigma \pi R^2} + j\omega \left(\frac{\mu \ln\left((d+a)/R\right)}{\pi} \frac{d}{a} + \frac{\mu}{4\pi} \right). \tag{4.31}$$

The real part of the low frequency impedance is equal to the dc resistance of the twin lead. The inductive term in (4.31) is the high frequency external impedance multiplied with d/a plus two times the internal impedance of the leads. At low frequency the model assumes surface currents are uniformly distributed around the periphery of the conductors. This spreading of currents increase the low frequency inductance by a factor of d/a. In reality the currents are uniformly distributed through out the cross section, which adds an internal inductance term, $\mu/8\pi$. The low frequency inductance calculated from conformal mapping is higher than the inductance calculated from an energy integral [28]. The main reason for this discrepancy is conformal mapping assumes high frequency magnetic field distribution independent of frequency. The difference between the real value and conformal mapping result depends on d/R ratio and is less than 10% for $d/R = 2$, and for $d/R > 6$ it is less than 1.5% [36].

Experimental results and calculations are plotted in Fig. 4.6. The resistance calculated from conformal mapping agrees well with the experimental data. For the inductive part, at low frequency there is the discrepancy mentioned above. But as the field penetration in the conductor lessens agreement improves. For the calculated data, resistance rises sooner than the experiment, due to higher low frequency inductance, since current crowding occurs sooner.

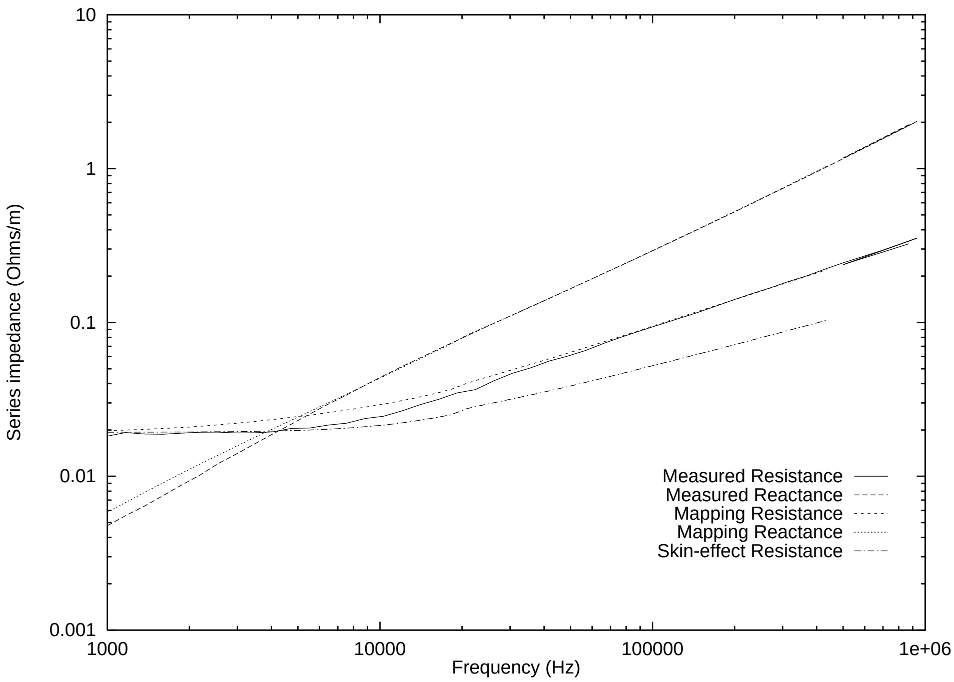

Figure 4.6: Experimental and computational results.

4.5.3 Rectangular Cylindrical Conductors

A practically more important structure is rectangular cylindrical conductors. For conductors with rectangular cross-section a conformal map is guaranteed to exist provided that there is at least one symmetry plane for the conductors, based on a Schwartz-Christoffel transformation [40]. A simple example for this case is co-planar strip lines. The map can be written for strips along the x-axis as

$$w = j\int_0^z \frac{dz'}{\sqrt{(z'^2 - d^2)(z'^2 - (w_s + d)^2)}}, \tag{4.32}$$

where w_s is the width of the conductors and d is the half the separation. The thickness of the conductor t is assumed to be much smaller than the width and separation. The scale factor is found by differentiation,

$$M(x) = \frac{1}{|\sqrt{(x^2 - d^2)(x^2 - (w + d)^2)}|}, \tag{4.33}$$

which is in the z-domain. For this case, it would be convenient to write down the total series impedance integral in the z-domain. To apply (4.20) to the map given by (4.32), the inverse function of (4.32) has to be found. Here we will scale the inductance to the z-domain from w-domain instead of scaling effective internal impedance from z-domain to w-domain. The inductance of an infinitesimal width in the mapped domain is

$$dL = \mu_0 \frac{|v_t - v_b|}{du}, \tag{4.34}$$

where $|v_t - v_b|$ is the plate separation and can be calculated by the integral

$$|v_t - v_b| = 2 \int_0^d \frac{dx}{\sqrt{(d^2 - x^2)((w + d)^2 - x^2)}}. \tag{4.35}$$

By substituting du in (4.34) with $M(x)dx$, the total series impedance expression can be written as

$$Z(\omega) = \left[2 \int_d^{d+w} \frac{dx}{j\omega\mu_0 |v_t - v_b|/M(x) + 2Z_{eii}} \right]^{-1}, \tag{4.36}$$

where Z_{eii} is the effective internal impedance of the strips, given by (3.9). The measured impedance per unit length and calculated results from (4.36) are shown in Fig. 4.7.

A more general result can be derived for thick, coplanar rectangular conductors. This class of structures can be mapped into parallel plates using two Schwartz-Christoffel maps. The first transformation maps the conductors into two thin, coplanar strip lines. Then using the map described above, the

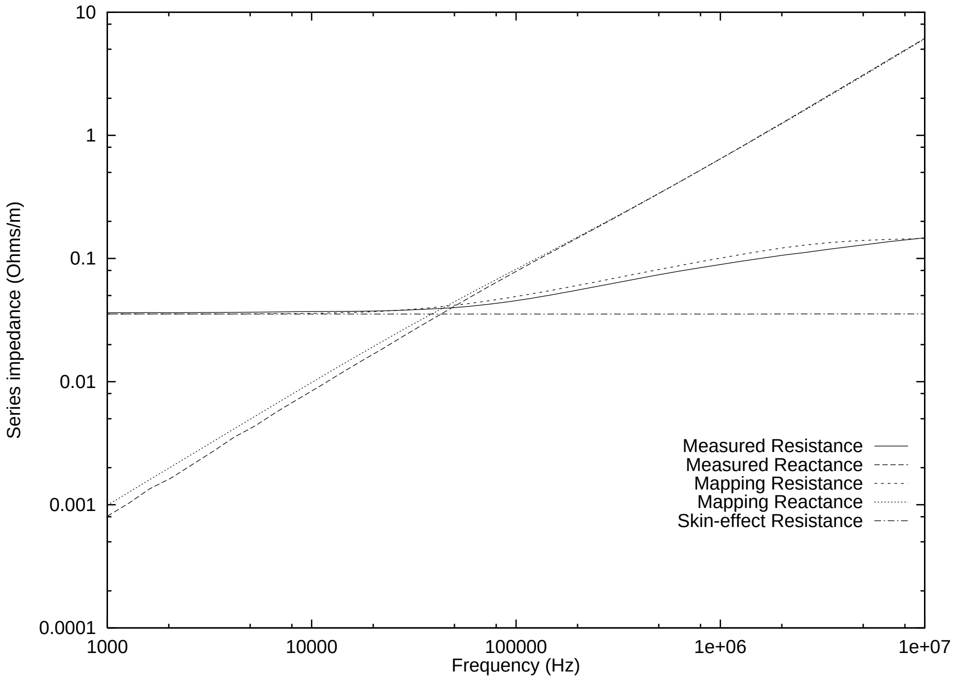

Figure 4.7: Comparison between measured (dotted lines) and calculated data for thin coplanar strip lines. The copper ($\sigma = 5.810^7$ $(\Omega\mathrm{m})^{-1}$) strips (13cm wide, 0.1cm apart, and 17μm thick) were supported by a 0.08cm thick ϵ_r=2.55 substrate.

strip lines can be mapped into parallel plates. The first Schwartz-Christoffel map is

$$z = c\int_0^{\xi}\sqrt{\frac{(\xi'^2 - 1/k_1^2)(\xi'^2 - 1/k_2^2)}{(\xi'^2 - 1)(\xi'^2 - 1/k^2)}}d\xi', \tag{4.37}$$

where the values of unknowns c, k, k_1, and k_2 are determined by the separation, width and thickness of the rectangular conductors. A program has been written using the steepest decent algorithm to solve for the unknowns. The second map required is similar to (4.32),

$$w = \int_0^{\xi}\frac{d\xi'}{\sqrt{(\xi'^2 - 1)(\xi'^2 - 1/k^2)}}. \tag{4.38}$$

In this case it is the easiest to scale the effective internal impedance from the z-plane into the ξ-plane and scale the inductance from the w-plane into the ξ-plane, which is a combination of previous examples. The scale factor from the z-plane to ξ-plane is

$$M(\xi) = \left|\frac{d\xi}{dz}\right| = \left|\frac{1}{c}\sqrt{\frac{(\xi^2 - 1)(\xi^2 - 1/k^2)}{(\xi^2 - 1/k_1^2)(\xi^2 - 1/k_2^2)}}\right|. \tag{4.39}$$

Also the scale factor from the w-plane to the ξ-plane can be written as

$$N(\xi) = \left|\frac{d\xi}{dw}\right| = \left|\sqrt{(\xi^2 - 1)(\xi^2 - 1/k_2^2)}\right|. \tag{4.40}$$

Using the scale factors above the total series impedance integral is given in the ξ-plane by

$$Z(\omega) = \left[\int_1^{1/k}\frac{d\xi}{j\omega\mu_0|v_t - v_b|N(\xi) + 2Z_{eii}M(\xi)}\right]^{-1}, \tag{4.41}$$

where $|v_t - v_b|$ is calculated in a similar fashion as the strip line case.

In (4.41), the effective internal impedance, Z_{eii}, should be represented in ξ-plane. This suggests finding an inverse map for the first Schwartz-Christoffel

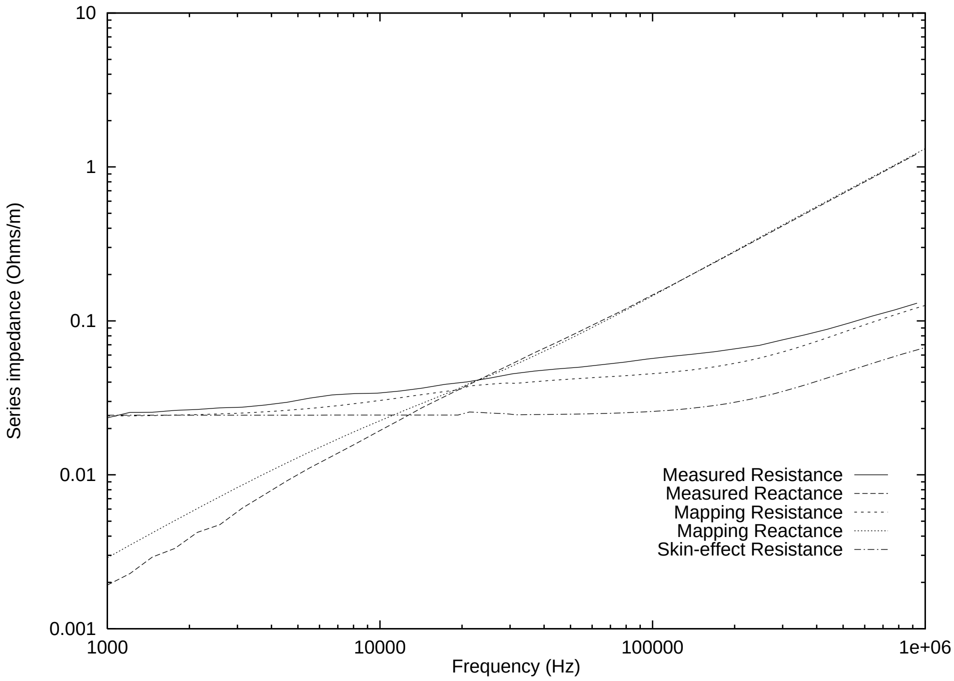

Figure 4.8: Real and imaginary parts of the impedance per unit length for coplanar, square hollow bars (0.40 cm sides, 0.038 cm wall thickness, 0.1 cm separation between bars, $\sigma = 1.510^7$ $(\Omega\text{m})^{-1}$ (brass)).

map (4.37). A way around this problem is to discritize the effective internal impedance, and solving for the mid points of each step in the ξ-plane. For the example presented here, this was not necessary because the conductors were brass tubes with thin walls and the Z_{eii} is assumed the same for all points on the surface. The effective internal impedance is calculated by setting $t/2$ to the wall thickness of the hollow brass conductor in (3.9). Results are shown in Fig. 4.8 in comparison with the measured data.

The conformal mapping technique was also applied to coplanar waveguide geometries with utmost success [19, 36]. The approach is numerically efficient, once the map parameters are found. The integrals are calculated using 24-point Gaussian Quadrature. This technique uses an isolated-conductor surface impedance in conjunction with a conformal map for the entire transmission line section to account for both skin-depth and current crowding effects. Using this technique, a circuit extraction of interconnect structures can be done efficiently. The extracted circuit would capture both skin and current crowding effects.

4.6 Practical Considerations

The new quasi-static technique proposed in this chapter is very efficient in calculating conductor loss in quasi-TEM transmission lines. The results obtained show excellent agreement with experimental measurements. This method divides the solution into two sections basically; first, it assigns an effective internal impedance value to the surface, calculated for an isolated conductor, which represents the inner parts of the conductors. Second, the conformal mapping technique is applied to solve for the external fields.

For conductors with rectangular cross-section, a Schwartz-Christoffel map can be used. The problem here is that it may not always be easy to obtain the map coefficients in (4.37). Once the coefficients are known for a given geometry, to calculate the series impedance is straightforward and very efficient. To overcome this difficulty a look up table of map coefficients can be created.

Another limitation surfaced as this method was tried on multi-conductor transmission lines. To calculate a suitable map with several conductors is very tedious and time consuming. Calculating the self and cross terms is not trivial using the conformal map [18]. A new method eliminates the limitations of conformal mapping, which uses the same approach of using the effective internal impedance of a conductor in conjunction with an external field solver.