Appendices

Appendix A

Non-Uniform Transmission Lines

Input impedance of an arbitrarily terminated non-uniform transmission line can be calculated using telegraphers' equations. Let us assume a transmission line along the z-axis with unit distance plate separation. For time the harmonic case, voltages and currents on the transmission line will satisfy these equations

$$\begin{aligned} V'' - \frac{Z'}{Z}V' - ZYV &= 0 && \text{(A.1)} \\ I'' - \frac{Y'}{Y}I' - ZYI &= 0 && \text{(A.2)} \\ V' &= -ZI && \text{(A.3)} \\ I' &= -YV, && \text{(A.4)} \end{aligned}$$

where V and I are voltages and currents on the transmission line and Z and Y are series impedance and shunt admittance at a given cross section of the line. Assuming no fringe fields, Z and Y are given by

$$\begin{aligned} Z &= \frac{\sqrt{j\omega\mu}}{w(z)} && \text{(A.5)} \\ Y &= w(z)\sqrt{\sigma + j\omega\epsilon}, && \text{(A.6)} \end{aligned}$$

where σ, μ, and ϵ are conductivity, permeability and permittivity of the surrounding medium respectively. ω is the frequency in rad/s and $w(z)$ is the width

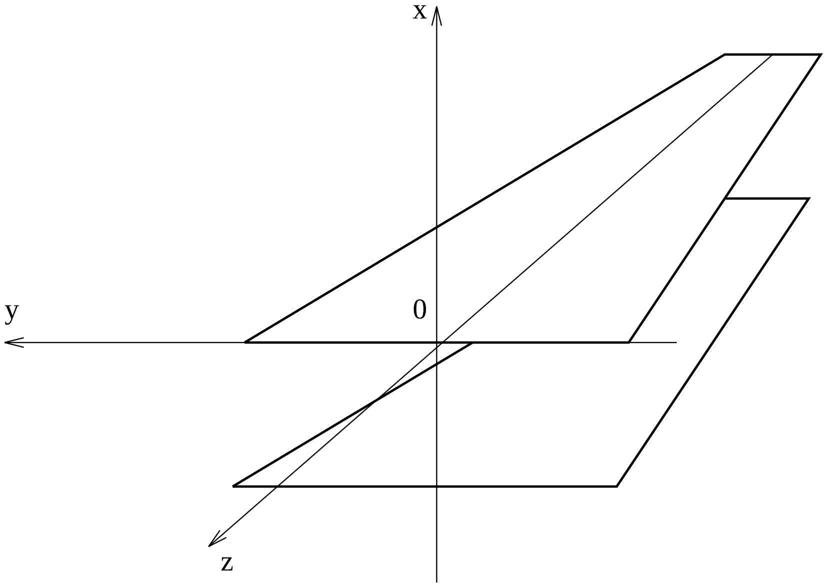

Figure A.1: A Non-uniform transmission line along z-axis, the width is changing along z-axis.

of the transmission line as a function of z. From the equations above,

$$\frac{Z'}{Z} = -\frac{w'(z)}{w(z)} \tag{A.7}$$

$$\frac{Y'}{Y} = \frac{w'(z)}{w(z)}. \tag{A.8}$$

For a transmission line structure with the width linear in z, $w(z)$ can be derived as

$$w(z) = \frac{w_o - w_i}{t} z, \tag{A.9}$$

where w_o and w_i are the width of the transmission line at the output and input ends respectively (Fig. A.2). With $w(z)$ given in (A.9), the general solution to V and I in (A.1) and (A.2) can be found as

$$V(z) = A\mathbf{H}_0^{(1)}(j\beta z) + B\mathbf{H}_0^{(2)}(j\beta z) \tag{A.10}$$

$$I(z) = Cz\mathbf{H}_1^{(1)}(j\beta z) + Dz\mathbf{H}_1^{(2)}(j\beta z), \tag{A.11}$$

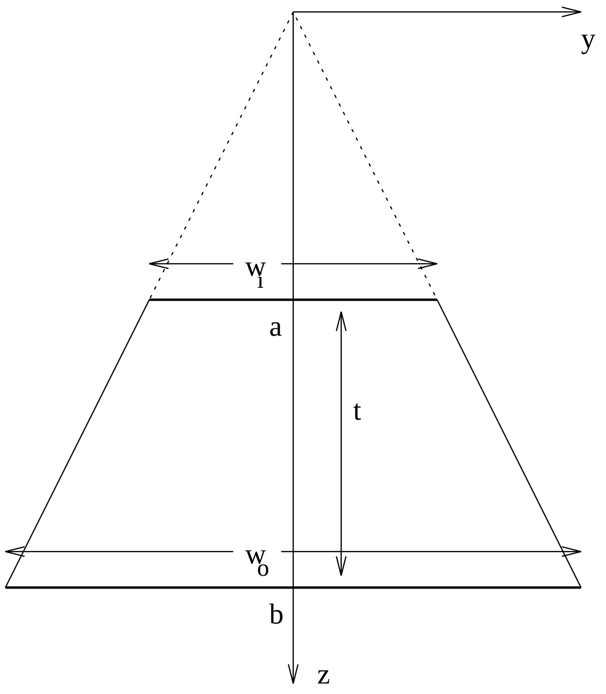

Figure A.2: Non-uniform transmission line looking from top, w_i and w_o are the widths at the input and output, respectively.

where $\mathbf{H}_n$ is the Henkel function of order n and $\beta^2 = ZY$. To find the input impedance at $z = a$, following boundary conditions are set

$$\begin{aligned} I(a) &= 1 \qquad (A.12) \\ V(b) &= Z_L I(b), \qquad (A.13) \end{aligned}$$

where Z_L is the termination impedance at $z = b$. Using (A.3), (A.4), and (A.11) $V(a)$ is solved. By setting $I(a) = 1$ Zin is found to be equal to $V(a)$. The resulting matrix equation is,

$$\begin{bmatrix} 0 & 0 & a\mathbf{H}_1^{(1)}(j\beta a) & a\mathbf{H}_1^{(2)}(j\beta a) \\ Y(a)\mathbf{H}_0^{(1)}(j\beta a) & Y(a)\mathbf{H}_0^{(2)}(j\beta a) & a\mathbf{H}_0^{(1)}(j\beta a) & a\mathbf{H}_0^{(2)}(j\beta a) \\ \mathbf{H}_0^{(1)}(j\beta b) & \mathbf{H}_0^{(2)}(j\beta b) & -Z_L b\mathbf{H}_1^{(1)}(j\beta b) & -Z_L b\mathbf{H}_1^{(2)}(j\beta b) \\ j\beta\mathbf{H}_1^{(1)}(j\beta a) & j\beta\mathbf{H}_1^{(2)}(j\beta a) & 0 & 0 \end{bmatrix} \times .$$

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -Z(a) \end{bmatrix} \tag{A.14}$$

Solving for the coefficients A and B, the input impedance at $z = a$ can be calculated;

$$Z_{in}(a) = A\mathbf{H}_0^{(1)}(j\beta a) + B\mathbf{H}_0^{(2)}(j\beta a). \tag{A.15}$$

Equation (A.15) is the equivalent impedance seen at the input with arbitrary termination. Using (A.15) cascading non-uniform transmission lines is possible.

Appendix B

Mutual Inductances of Ribbons

The low frequency inductance of transmission lines can be calculated from energy expressions since the current distribution is known and set by the resistance[35, 9]. For the ribbon technique self and mutual inductances between ribbons are required. For the practical interconnect geometries the ribbons are either parallel or perpendicular to each other but here a general mutual inductance expression will be derived for arbitrary placed ribbons.

The stored magnetic energy per unit length, in terms of the inductance per unit length is

$$W_m = \frac{1}{2} L I^2, \tag{B.1}$$

where I is the total current. The stored magnetic energy can also be calculated from the magnetic vector potential and the current distribution. The magnetic vector potential is written in terms of current distribution for two dimensional structures[1]

$$\vec{A}(\vec{r}) = -\frac{\mu_0}{2\pi} \int \vec{J}(\vec{r'}) \log(|\vec{r} - \vec{r'}|) ds', \tag{B.2}$$

where $\vec{r}$ and $\vec{r'}$ are the observation and source coordinates, respectively, and $\vec{J}$

---

[1] Total current throughout the cross section is assumed to be zero, e.i., there is no current return at infinity.

is the current density. The stored magnetic energy is

$$\begin{aligned} W_m &= \frac{1}{2}\int \vec{J}(\vec{r})\cdot\vec{A}(\vec{r})ds \\ &= -\frac{\mu_0}{2\pi}\int \vec{J}(\vec{r})\left(\int \vec{J}(\vec{r'})\log(|\vec{r}-\vec{r'}|)ds'\right)ds \\ &= -\frac{\mu_0}{2\pi}\int\int \vec{J}(\vec{r})\vec{J}(\vec{r'})\log(|\vec{r}-\vec{r'}|)ds'ds. \end{aligned} \tag{B.3}$$

Equating the above equation to (B.1) an expression for inductance is obtained,

$$L = -\frac{\mu_0}{2\pi I^2}\int\int \vec{J}(\vec{r})\vec{J}(\vec{r'})\log(|\vec{r}-\vec{r'}|)ds'ds, \tag{B.4}$$

where I is the total current.

Since the current distribution is uniform on ribbons, The $\vec{J}$'s can be replaced by the total current divided by the ribbon widths in (B.4). In a multiconductor transmission line (B.4) can be used to calculate partial inductances between the ribbons k and m

$$L_{km} = -\frac{\mu_0}{2\pi w_k w_m}\int_{\ell_k}\int_{\ell_m}\log(|\vec{r}-\vec{r'}|)dl_k dl_m, \tag{B.5}$$

where w's are the widths of ribbons and ℓ's are the paths along the width of the ribbons.

The double integral in (B.5) can be evaluated using the complex calculus. For a complex z,

$$\log(|z|) = \Re\left\{\log(z)\right\}. \tag{B.6}$$

The integral in (B.5) is evaluated in the complex plane as

$$\int\int \log(z)dzdz = \frac{z^2}{2}\left(\log(z)+1.5\right). \tag{B.7}$$

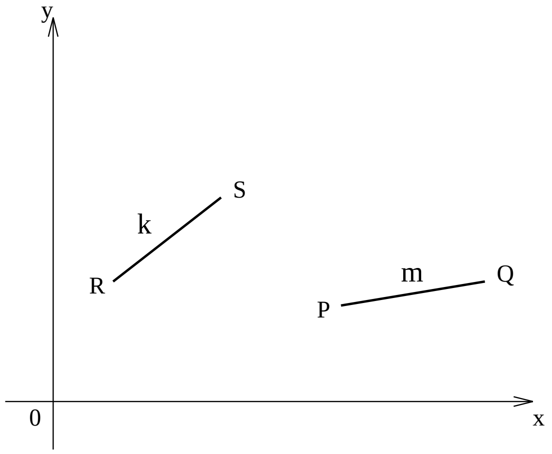

Figure B.1: Two ribbons for mutual inductance calculation. $R = (x_{1k}, y_{1k})$, $S = (x_{2k}, y_{2k})$, $P = (x_{1m}, y_{1m})$, and $Q = (x_{2m}, y_{2m})$. Current flows in a direction normal to the plane of the page.

By mapping the transverse plane of transmission line into the complex plane, the partial inductance is calculated

$$L_{km} = \frac{\mu_0}{2\pi w_k w_m} \Re \left\{ \frac{(z_k - z_m)^2}{2} \left(\log(z_k - z_m) + 1.5\right) \Big|_{z_m = x_{1m} + j y_{1m}}^{x_{2m} + j y_{2m}} \Big|_{z_k = x_{1k} + j y_{1k}}^{x_{2k} + j y_{2k}} \right\}, \tag{B.8}$$

where (x_1, y_1) and (x_2, y_2) are the coordinates of the edges of ribbons (Fig. B.1).

Using the above equation the ribbon technique can be applied to not only rectangular bars but also to conductors with polygonal cross sections. The difficulty here is to calculate an effective internal impedance for conductors with polygonal cross section.