Efficient Calculation of Surface Impedance for Rectangular Conductors

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Abstract

We have developed a new expression for the frequency-dependent surface impedance of a rectangular bar that is easily used, and is numerically efficient. By dividing the metal bar into rectangular and square sections, skin depth-induced current crowding to the surfaces and corners can be accurately modeled. Comparison to measured data shows excellent agreement over a wide frequency range, covering the transition from dc-like behavior to skin-depth limited behavior.

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Introduction: Recently a new, simple technique for the evaluation of conductor loss in quasi-TEM transmission lines has been demonstrated [1, 2]. Efficient use of this conformal mapping technique requires the knowledge of the specific surface impedance of each conductor in the transmission line. For a conductor with circular cross-section, it is possible to solve the Helmholtz equation exactly. However, many transmission lines make use of conductors with rectangular cross-section. There are a variety of techniques that have been developed for use with rectangular cross-sections, but many are numerically intensive. In this letter we show a new, numerically efficient technique that accurately predicts the surface impedance of rectangular bars ranging in aspect ratio from a square bar to a wide flat plate.

Model: A very simple approximation for the specific surface impedance Z_S (in Ω/square) of a "flat" conductor that is much wider than it is thick (i.e., a "thin" conducting plate) is [3]

$$Z_S^{plate} = \frac{\sqrt{\frac{\omega\mu}{2\sigma}}(1+j)}{\mathbf{tanh}\left\{\sqrt{\frac{\omega\mu\sigma}{2}}\left(\frac{t}{2}\right)(1+j)\right\}} = \frac{\frac{1}{\sigma\delta}(1+j)}{\mathbf{tanh}\left\{\frac{1}{2}\left(\frac{t}{\delta}\right)(1+j)\right\}} \qquad (1)$$

where t is the thickness and σ is the conductivity of the conductor, ω is the angular frequency, μ is the permeability of the metal, and δ is the skin depth in the conductor. Note that since the plate has two surfaces, eq. 1 must be used for each side of the conductor. In time domain simulations it is quite important that the surface impedance have the correct dc behavior. The appropriate dc limit of eq. 1 should be $2R_S$ (the factor of two appearing because this resistance represents only one side of the plate), where R_S is the dc sheet resistance of the entire plate (given by $1/\sigma t$). It is easily verified that Eq. 1 does produce the appropriate low frequency limit, i.e., as $\omega \to 0$, $Z_S \to 2/\sigma t$.

For rectangular conductors with appreciable thickness, however, the simple result given by eq. 1 will not hold. An accurate approximate expression for a thick conducting plate should yield both the correct dc resistance of the conductor as $\omega \to 0$ as well as the proper skin effect behavior at high frequency. The simplest approach is to divide the rectangular cross section into segments, as shown in Fig. 1. Note that the frequency dependence is set by the ratio of the skin depth to the "thickness" of the conducting region. For the rectangular regions (segments *A* in Fig. 1) this is just half the thickness *t*, so the rectangular sections should have a surface impedance given by eq. 1.

In the square corner sections (segments *C* in Fig. 1) the skin effect will cause current crowding toward the outside corner of the region. By symmetry, we need only find the surface impedance of half of the square region, divided along a diagonal (region *C′* in Fig. 1). To capture this effect, consider Fig. 2 for region *C′* of the conductor. We divide the region into *N* triangular patches, and use the distance from the base of each patch to the inside corner, h_n, as the "thickness" of this segment of conductor. For the n^{th} (n = 0, 1, 2, ..., N-1) segment in Fig. 2, the height h_n is given by

$$h_n = \frac{t}{2}\sqrt{1+\left(\frac{n+\frac{1}{2}}{N}\right)^2} \tag{2}$$

and the width w_n by

$$w_n = \frac{h_n}{2}\left[\frac{1}{N+\left(n+\frac{1}{2}\right)\frac{n}{N}}+\frac{1}{N+\left(n+\frac{1}{2}\right)\frac{n+1}{N}}\right] \tag{3}$$

For a triangular section, the surface impedance expression can be found by applying transverse resonance and non-uniform transmission line analysis [4, 5]. The total input impedance for a triangular "transmission line" with width w_n (at the "input" end, or front surface), "plate" separation of one unit distance, and length h_n, filled with a uniform conducting material of conductivity σ, assuming an open circuit termination, is

$$Z_{in} = \frac{j\sqrt{j\omega\mu\sigma}}{w_n\sigma}\frac{\mathbf{J_0}\left(j\sqrt{j\omega\mu\sigma}\,h_n\right)}{\mathbf{J_1}\left(j\sqrt{j\omega\mu\sigma}\,h_n\right)} \tag{4}$$

where $\mathbf{J}_0$ and $\mathbf{J}_1$ are Bessel functions of the first kind. The approximate surface impedance Z_S^n of the n^{th} patch is obtained by applying eq. 4 and normalizing to the width of the original region $\frac{t}{2N}$

$$Z_S^n = \frac{j\sqrt{j\omega\mu\sigma}}{w_n\sigma}\frac{\mathbf{J_0}\left(j\sqrt{j\omega\mu\sigma}\,h_n\right)}{\mathbf{J_1}\left(j\sqrt{j\omega\mu\sigma}\,h_n\right)}\frac{t}{2N} \tag{5}$$

This expression is the desired approximation for the surface impedance as a function of position on the corners of the conductor.

Results and Conclusions: In actual experiment, it is the total internal impedance per unit length of the conductor Z_{tot}, rather than the surface impedance, that can be measured. The total internal impedance is the parallel sum of the surface impedances of the various regions, given by

$$Z_{tot} = \left\{ 2 \cdot \left[\frac{Z_S^{plate}}{(W - t)} \right]^{-1} + 8 \cdot \sum_{n=0}^{N-1} \left[\frac{Z_S^n}{(t/2N)} \right]^{-1} \right\}^{-1} \qquad (6)$$

To check the validity of our approximation, we consider the total internal impedance of a square conductor, which should be the most severe test of our technique. Figure 3 shows a comparison between eq. 6 and the experimental results of Haefner [6], for square conductors. This data has also been used as a test case by ref. [7]. As shown, even for a very small number of segments, the agreement is excellent. Using only one segment, the error is not worse than 5% at any frequency, and for four segments the error is never more than 0.2%.

In summary, we have developed a new approximation for the frequency-dependent surface impedance of thick rectangular bars. The expressions are easily used, and are numerically efficient. These results should be useful in the calculation of conductor loss in planar transmission lines and interconnects, providing an accurate transition from dc-like behavior to skin-depth limited behavior.

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References:

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Figure Captions

Figure 1: Thick rectangular conductor. The surface impedance of regions A is given approximately by eq. 1, while regions C' are given by eq. 5.

Figure 2: n^{th} triangular patch in the corner region C' of Fig. 1. The triangular transmission line used for input impedance calculation has "length" h_n and width w_n .

Figure 3: Comparison of calculated and measured total internal resistance of a square metal bar. ⊙: measured data from ref. [Haefner, 1937 #17]; solid lines: eq. 6.

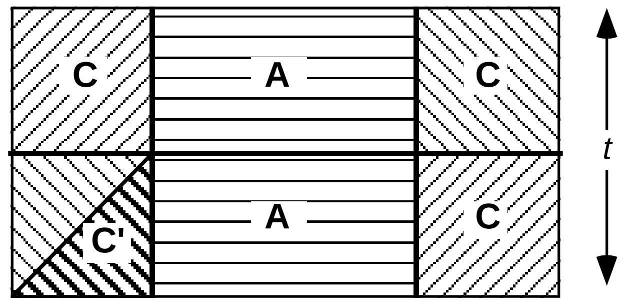


Figure 1

Tuncer "Efficient Calculation of Surface Impedance for Rectangular Conductors"

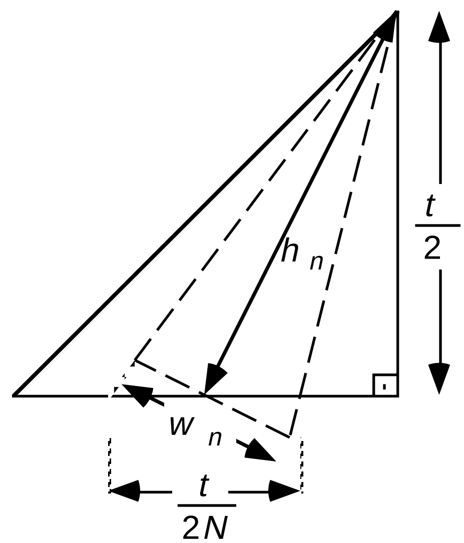
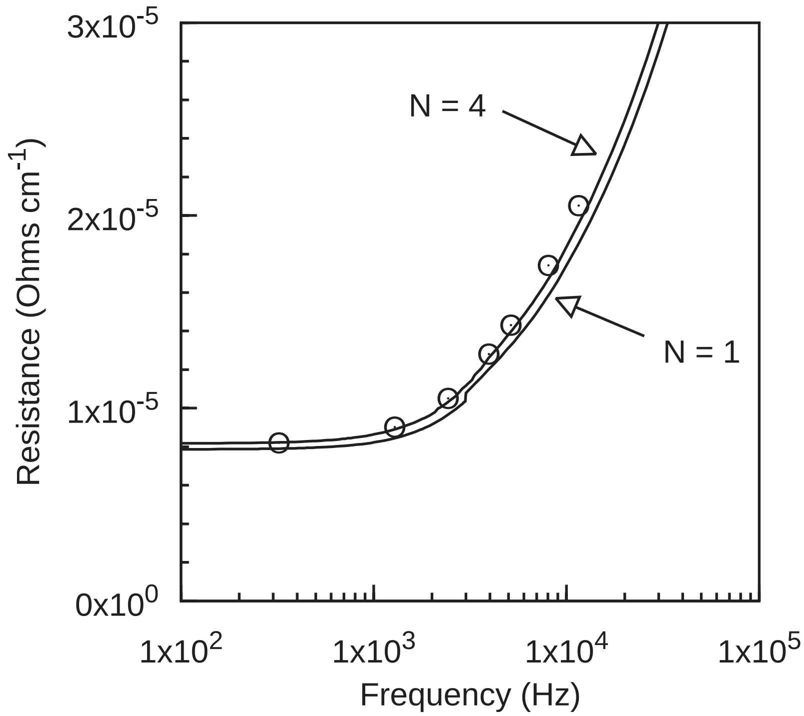


Figure 2

Tuncer "Efficient Calculation of Surface Impedance for Rectangular Conductors"



3x10^-5
2x10^-5
1x10^-5
0x10^0
Resistance (Ohms cm^-1)
N = 4
N = 1
1x10^2
1x10^3
1x10^4
1x10^5
Frequency (Hz)