Silicon as a mechanical material

- **classic reference in the field:**
  - **K.E. Petersen "Silicon as a Mechanical Material", *Proceedings of the IEEE*, Vol. 70, No.5, May 1982.**
    - **http://robotics.eecs.berkeley.edu/~tahhan/MEMS/petersen/mems_petersen.htm**
  - **tenants:**
    - **silicon is abundant, inexpensive, and can be produced in extremely high purity and perfection;**
    - **silicon processing based on very thin deposited films which are highly amenable to miniaturization;**
    - **definition and reproduction of the devices, shapes, and patterns, are performed using photographic techniques that have already proved of being capable of high precision;**
    - **silicon microelectronic (and therefore also mems) devices are batch-fabricated.**

Thermal properties

- **thermal diffusion equation**

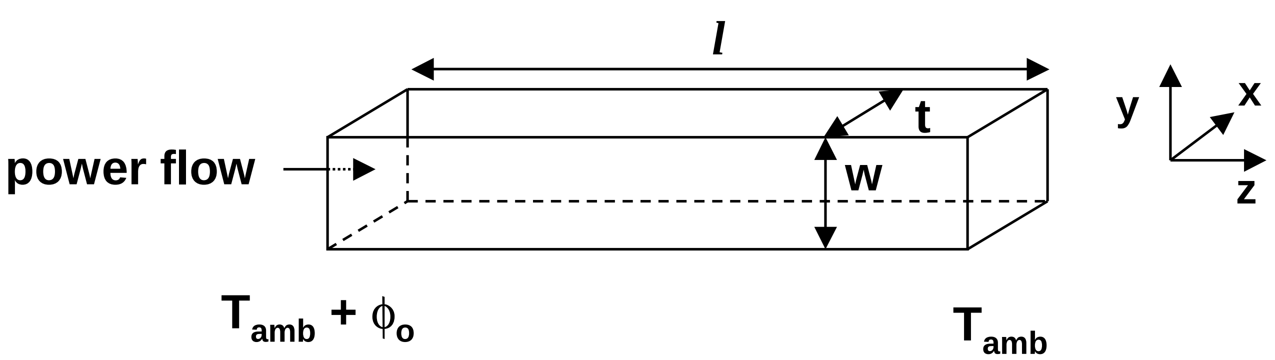
$$\nabla^2\phi = \frac{\rho \cdot C}{\kappa}\frac{\partial\phi}{\partial t}$$

  - ϕ **is temperature change relative to ambient**
  - ρ **is material density**
  - **C is specific heat (units: energy · mass^{-1} · kelvin^{-1})**
  - κ **is thermal conductivity (units: power · distance^{-1} · kelvin^{-1})**
- **if ϕ was a voltage this looks a lot like an electrical problem…**
- **or if ϕ was a concentration this looks a lot like a diffusion problem for constant thermal diffusivity D = κ/ρC**

$$\frac{\partial\phi}{\partial t} = \frac{\kappa}{\rho \cdot C}\nabla^2\phi$$

1-d (uniform) heat flow problem

- heat flow along z-axis, uniform in x-y plane

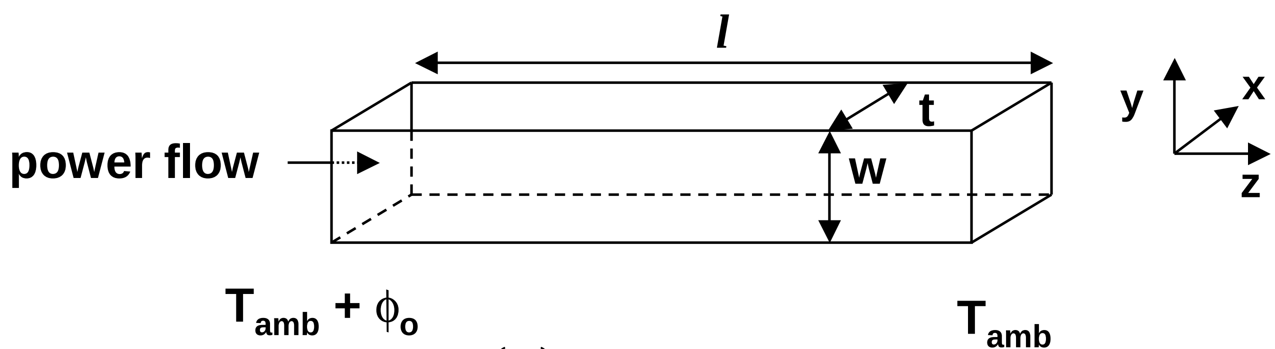


$$\underbrace{\nabla^2 \phi}_{1-d} = \frac{\rho \cdot C}{\kappa} \frac{\partial \phi}{\partial t} \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial z^2} = \frac{\rho \cdot C}{\kappa} \frac{\partial \phi}{\partial t}$$

$$= \frac{\partial^2 \phi}{\partial z^2}$$

1-d "sinusoidal" heat flow

- **let's assume the power flowing from the "source" end of the bar is sinusoidally varying in time**



$$\phi(z) = \phi_o\, e^{j\cdot\omega\cdot t - \gamma\cdot z}$$

$$\frac{\partial^2\phi}{\partial z^2} = \frac{\rho\cdot C}{\kappa}\frac{\partial\phi}{\partial t} \quad\Rightarrow\quad \gamma^2\cdot\phi_o\, e^{j\cdot\omega\cdot t - \gamma\cdot z} = \frac{\rho\cdot C}{\kappa}\cdot(j\omega)\cdot\phi_o\, e^{j\cdot\omega\cdot t - \gamma\cdot z}$$

$$\Rightarrow\quad \gamma = \sqrt{\frac{j\cdot\omega\cdot\rho\cdot C}{\kappa}}$$

Thermal diffusion length

- so form of 1-d solution to thermal diffusion equation is

$$\phi = \phi_o \, e^{j \cdot \omega \cdot t - \gamma \cdot z}$$

- the "complex thermal propagation constant" is

$$\gamma = \sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}}$$

- units: { [sec^{-1} · mass · distance^{-3} · energy · mass^{-1} · kelvin^{-1}] / [energy · sec^{-1} · distance^{-1} · kelvin^{-1}] }^{½}

  = (distance)$^{-1}$

  – thermal diffusion length is just 1 / γ

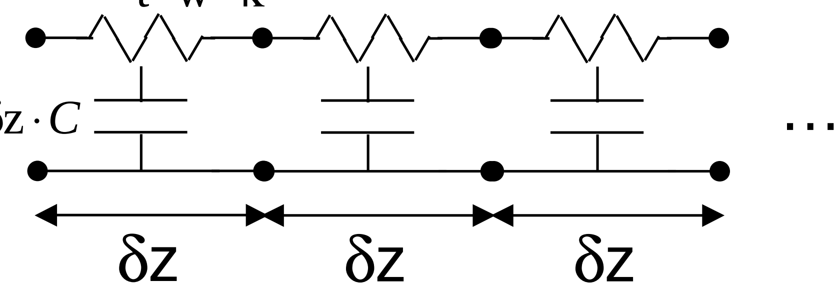
$$L_{thermal} = \sqrt{\frac{\kappa}{j \cdot \omega \cdot \rho \cdot C}}$$

Thermal "transmission line"

- **this is the same solution you would get from solving a distributed circuit that looks like**

$$R = \frac{\delta z}{t \cdot w \cdot \kappa}$$

$$C = \rho \cdot t \cdot w \cdot \delta z \cdot \mathcal{C}$$



- **to solve need the "per unit length series impedance" Z and "per unit length shunt admittance" Y, then γ = (Z•Y)½**

$$Z = \frac{1}{t \cdot w \cdot \kappa}$$

$$Y = (j\omega) \cdot \rho \cdot t \cdot w \cdot C$$

$$\gamma = \sqrt{\frac{(j\omega) \cdot \rho \cdot t \cdot w \cdot C}{t \cdot w \cdot \kappa}} = \sqrt{\frac{(j\omega) \cdot \rho \cdot C}{\kappa}}$$

thermal resistance of a rectangular bar

- **thermal – to – electrical analogy**
  - **thermal "input" power (heat) P : current**
  - **temperature rise/drop ΔT (ϕ): voltage**
  - **Z: thermal impedance per unit length** $$Z = \frac{1}{t \cdot w \cdot \kappa}$$
  - **C: thermal capacitance per unit length** $$Y = (j\omega) \cdot \rho \cdot t \cdot w \cdot C$$
- **what is the "thermal input impedance" $Z_{thermal}$ of a rectangular bar *l* long, w wide, and t thick, one end connected to a perfect heat sink?**
  - **from t-line analogy:** $$Z_{thermal} = Z_{in}(l) = Z_o \cdot \frac{Z_L + Z_o \cdot \tanh(\gamma \cdot l)}{Z_o + Z_L \cdot \tanh(\gamma \cdot l)}$$
  - **from t-line analogy $Z_o = (Z/Y)^{1/2}$**
  - **for a perfect heat sink "load" $Z_L = 0$ (i.e., a short!)**
- **what is the temperature rise at the "input" end?**
  - **from analogy:**

$$\phi_{end} = P \cdot Z_{thermal} = P \cdot \sqrt{\frac{Z}{Y}} \cdot \frac{0 + \sqrt{Z/Y} \cdot \tanh\left(\sqrt{Z \cdot Y} \cdot l\right)}{\sqrt{Z/Y} + 0 \cdot \tanh\left(\sqrt{Z \cdot Y} \cdot l\right)} = \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{j \cdot \omega \cdot \kappa \cdot \rho \cdot C}} \cdot \tanh\left(\sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l\right)$$

What does this really mean?

- **consider "low frequency": length of bar much less than thermal diffusion length**

$$\gamma \cdot l = \frac{l}{L_{thermal}} = \sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l << 1$$

$$\phi_{end} = \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{j \cdot \omega \cdot \kappa \cdot \rho \cdot C}} \cdot \underbrace{\tanh\left(\underbrace{\sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l}_{<<1} \right)}_{\approx \text{argument}} \approx \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{j \cdot \omega \cdot \kappa \cdot \rho \cdot C}} \cdot \sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l$$

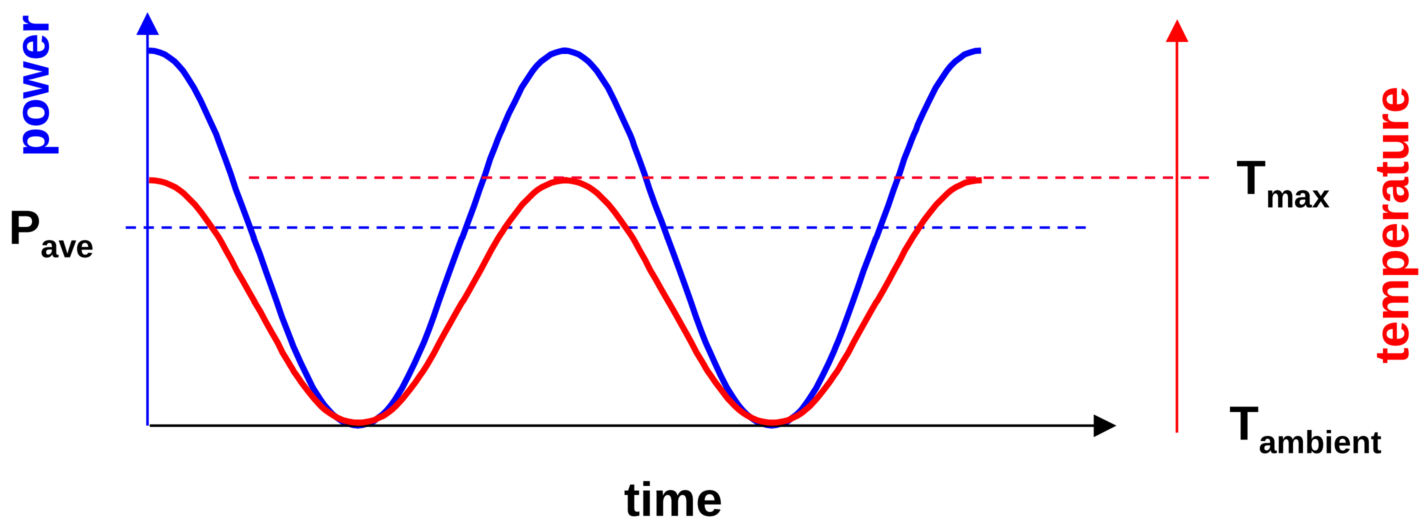
$$\phi_{end}^{\text{"low freq"}} \approx \frac{P}{\underbrace{t \cdot w}_{\text{power density}}} \cdot \frac{l}{\kappa}$$

- **looks just like the resistance of a bar with "conductivity"** κ

Low frequency thermal response

$$\Delta T = \frac{l}{t \cdot w \cdot \kappa} P$$

- the actual input power looks like
  - $P_{in} = P_{ave} \bullet [\, 1 + \cos(\omega t)\,]$
- at "low frequency" the temperature "follows" (in phase) the input power
  - temperature rises as power increases
  - temperature falls as power decreases



High frequency thermal response

- **consider "high frequency": length of bar much longer than thermal diffusion length**

$$\gamma \cdot l = \frac{l}{L_{thermal}} = \sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l >> 1$$

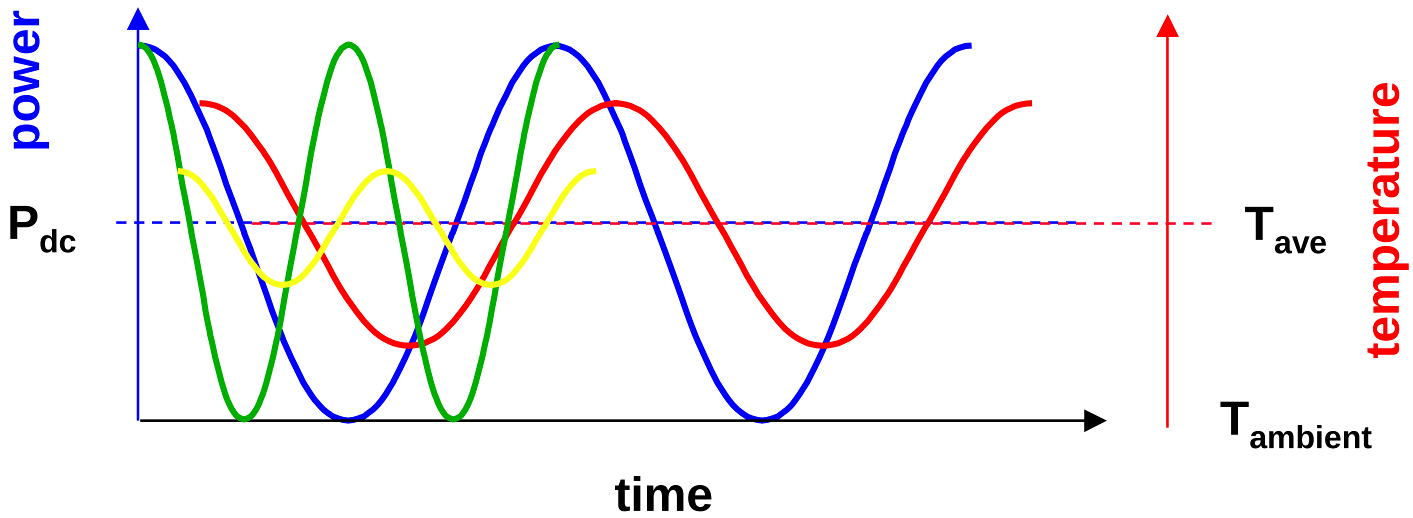
$$\phi_{\mathrm{end}} = \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{j \cdot \omega \cdot \kappa \cdot \rho \cdot C}} \cdot \underbrace{\tanh\left(\underbrace{\sqrt{\frac{j \cdot \omega \cdot \rho \cdot C}{\kappa}} \cdot l}_{>>1}\right)}_{\approx 1} \approx \frac{P}{t \cdot w} \cdot \underbrace{\sqrt{\frac{1}{j \cdot \omega \cdot \kappa \cdot \rho \cdot C}}}_{\mathrm{L_{thermal}}/\kappa}$$

$$\phi_{\mathrm{end}}^{\text{"high freq"}} \approx \frac{P}{\underbrace{t \cdot w}_{\text{power density}}} \cdot \frac{\mathrm{L_{thermal}}}{\kappa} = \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{2\omega\kappa\rho C}} \cdot (1 - j)$$

High frequency thermal response

$$\phi_{\text{end}}^{\text{"high freq"}} \approx \underbrace{\frac{P}{t \cdot w}}_{\text{power density}} \cdot \frac{L_{\text{thermal}}}{\kappa} = \frac{P}{t \cdot w} \cdot \sqrt{\frac{1}{2\omega\kappa\rho C}} \cdot (1 - j)$$

- the temperature "lags" the power by 45°
- as the frequency increases the magnitude of the temperature change decreases as $\sqrt{\omega}$



Thermal "spreading resistance" of a semi-infinite medium

- **Assume heated body is spherical, radius r_o**
  - **time harmonic solution is**

$$\phi = \phi_o \frac{r_o}{r} e^{j\cdot\omega\cdot t - \frac{(r - r_o)}{L_{thermal}}} \qquad L_{thermal} = \sqrt{\frac{\kappa}{j\cdot\omega\cdot\rho\cdot C}}$$

  - **dc result for temperature rise at surface of sphere of radius r_o, heat sink at infinity, input power P**

$$\Delta T = \underbrace{\frac{1}{2\pi\cdot\kappa\cdot r_o}}_{\substack{\text{thermal spreading} \\ \text{resistance}}} P$$

Time domain thermal solutions

- **recall the thermal diffusion equation is** $$\frac{\partial \phi}{\partial t} = \frac{\kappa}{\rho \cdot C} \nabla^2 \phi$$
- **or using** $$D = \frac{\kappa}{\rho \cdot C}$$
- **we have the conventional diffusion equation**

$$\frac{\partial \phi}{\partial t} = D \cdot \nabla^2 \phi$$

  - **now we can use boundary/initial value solutions from general diffusion theory**

Boundary/initial conditions: constant temperature source diffusion

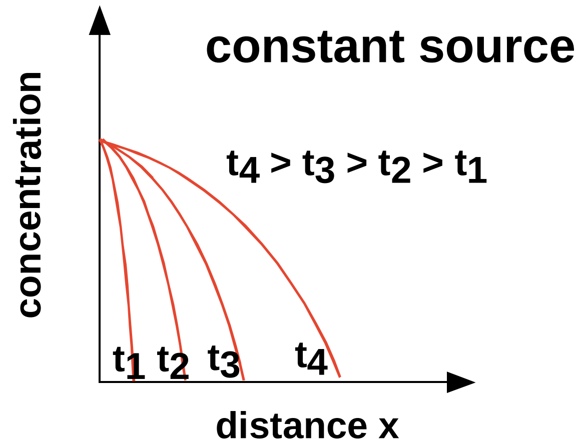
- example: at $t = 0$, at $x = 0$ the temperature of a semi-infinite slab is set to temperature T_o above the ambient temperature T_a
  - recall $\phi(x, t) = \Delta T = T - T_a$
- initial conditions
  - $\phi(x, 0) = 0$ (sample temperature initially constant at ambient)
- boundary conditions
  - $\phi(0, t) = \phi_S$, a constant
  - $\phi(\infty, t) = 0$ (sample is infinitely thick)
- solution is a complimentary error function

$$\operatorname{erfc}(y) = 1 - \frac{2}{\sqrt{\pi}} \int_0^y \exp\left(-z^2\right) dz \qquad \phi(x, t) = \phi_s \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right)$$

  - note argument is $x/2\sqrt{Dt}$ $$\frac{x}{2\sqrt{D \cdot t}} = \frac{x}{2\sqrt{\frac{\kappa}{\rho \cdot C} \cdot t}}$$

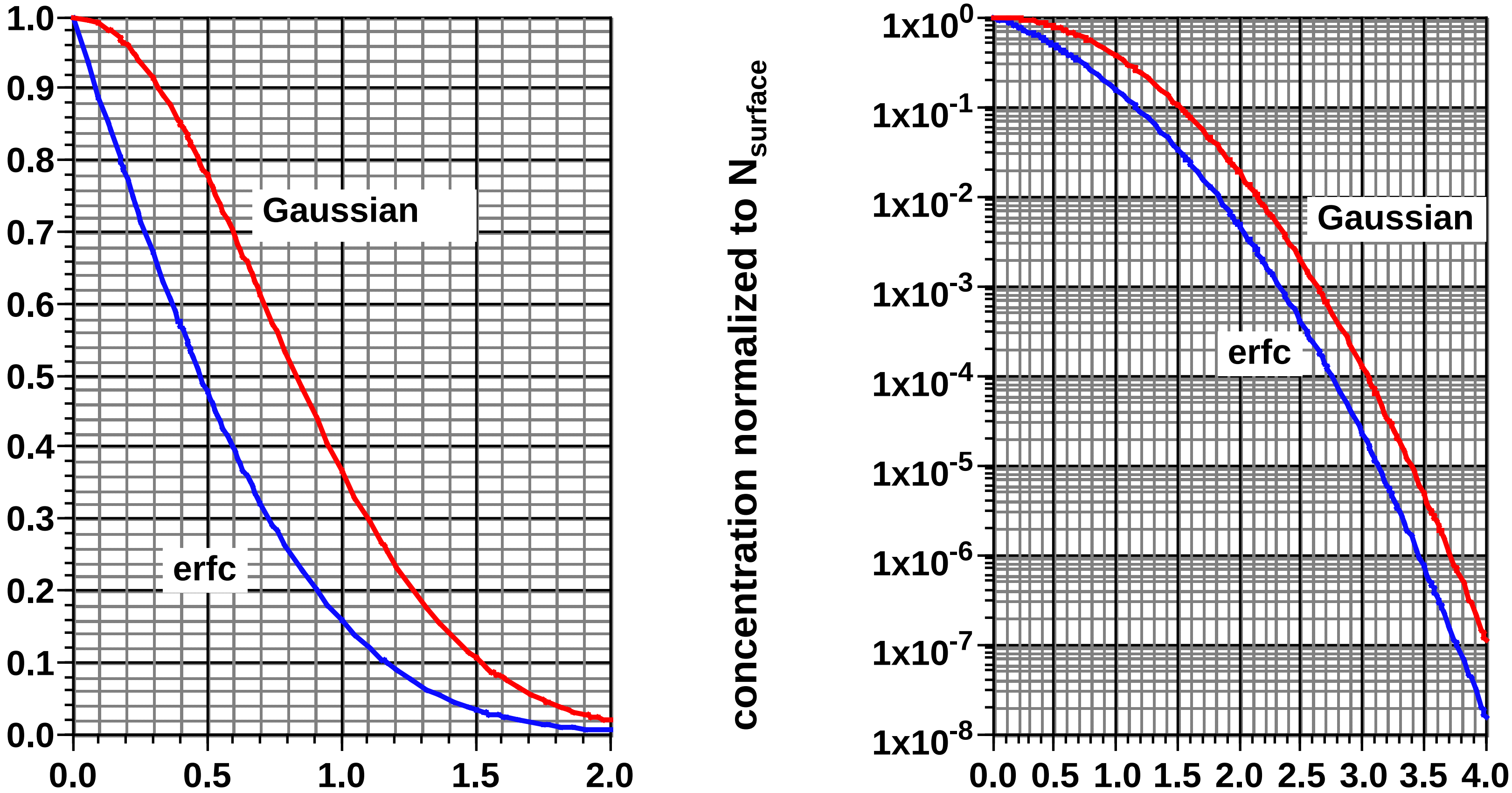
Constant diffusivity results

- solutions to the diffusion equation
  - "constant source"
    - "unlimited supply" of thermal power
    - erfc shape



  - diffusion length: $$l \approx \sqrt{D \cdot t} = \sqrt{\frac{\kappa}{\rho \cdot C} \cdot t}$$

gaussian and erfc profiles



argument “z” = $x / 2\sqrt{Dt}$

Thermal expansion and thermal property values

- **temperature coefficient of thermal expansion, TCE, α**
  - **for sample of length L, fractional change in length for change in temperature**

$$\alpha = \frac{1}{L} \cdot \left(\frac{\partial L}{\partial T} \right)_{\text{constant } P}$$

	Density (g/cm^3)	Thermal Conductivity (W/cm°C)	Thermal Expansion (10^{-6}/°C)	Specific Heat (J / g K)
Diamond	3.5	20	1.0	0.50
Al_2O_3	4.0	0.5	5.4	0.77
Iron	7.8	0.803	12	0.4
SiO_2	2.5	0.014	0.55	0.74
Si	2.3	1.57	2.33	0.71
Al	2.7	2.36	25	0.88

Mechanical properties

- consider elastic media: “Hooke’s law” applies
  - restoring force is proportional to displacement
- consider a bar under longitudinal tension or compression
  - under tension
    - length increases
    - cross sectional area decreases
    - note TOTAL volume can increase or decrease, depending on material constants!
- relation between stress and strain
  - stress (longitudinal) = force per unit area (units of pressure!)
  - strain: fractional change in length $\delta L/L$ (dimensionless)
  - Young’s modulus E = stress / strain (units of force per area)
    - i.e.,

$$\text{stress} = E \cdot \frac{\delta L}{L}$$

Young’s modulus is the stress you would have to apply to double the length of the bar (I.e., $\delta L = L$)

http://www.britannica.com/seo/y/youngs-modulus/

Poisson's ratio

- **consider a bar under longitudinal tension or compression**
  - **under tension**
    - **length increases: Young's modulus**
    - **ALSO: cross sectional area decreases**
    - **this constitutes a transverse strain $\delta W/W$**
  - **Poisson's ratio ν = transverse strain / longitudinal strain**
    - **dimensionless (since both strains are dimensionless)**

$$\nu = \left(\frac{\delta W}{W}\right) \Big/ \left(\frac{\delta L}{L}\right)$$

$$\frac{\delta W}{W} = \frac{\nu}{E} \cdot (\text{longitudinal stress})$$

overall volume change

assume bar subjected to longitudinal tensile stress

$$V \propto (L+\delta L)\cdot(W-\delta W)^2 = (L+\delta L)\cdot\left(W^2 - 2W\cdot\delta W + \underbrace{[\delta W]^2}_{2\text{nd order}\approx 0}\right)$$

$$\approx (L+\delta L)\cdot\left(W^2 - 2W\cdot\delta W\right) = \underbrace{L\cdot W^2}_{\propto\text{ unstrained volume}} - 2W\cdot\delta W\cdot L + \delta L\cdot W^2 - 2W\cdot\underbrace{\delta W\cdot\delta L}_{2\text{nd order}\approx 0}$$

$$\approx \underbrace{L\cdot W^2}_{\propto\text{ unstrained volume}} - 2W\cdot\delta W\cdot L + \delta L\cdot W^2 = L\cdot W^2 + W^2\cdot\delta L\cdot\left(1 - 2\cdot\frac{\delta W}{W}\cdot\frac{L}{\delta L}\right)$$

$$= L\cdot W^2 + \overbrace{W^2\cdot\delta L\cdot\underbrace{(1 - 2\cdot\nu)}_{\substack{<0 \text{ if } \nu>0.5 \\ >0 \text{ if } \nu<0.5}}}^{\text{volume change}}$$

- **ν > 0.5: total volume DECREASES under longitudinal tensile stress**
- **ν < 0.5: volume INCREASES**

Young’s modulus and Poisson’s ratio of “common” materials

- units
  - 10^6 pounds per square inch (psi) = mega-psi
    = 6.89×10^9 Newton/m^2 = 6.89 gigaPascal
- is temperature dependent

material	Young’s modulus (@ 300K) (GigaPascal)	Poisson’s ratio
diamond	1000	0.067
silicon	200	0.21
SiO2	70	0.17
Al2O3 (sapphire)	500	0.23
Iron	200	
Aluminum	70	0.34

Static beam equations

- **simple beam L long, w wide, t thick**
  - **beam: L >> w and t**
- **cantilever beam: supported at one end only**
  - **point force F at position a**
  - **displacement y at position x**

$$y(x) = \frac{F}{6 \cdot E \cdot I} \cdot \begin{cases} x^2 \cdot (3 \cdot a - x) & x < a \\ a^2 \cdot (3 \cdot x - a) & x > a \end{cases}$$

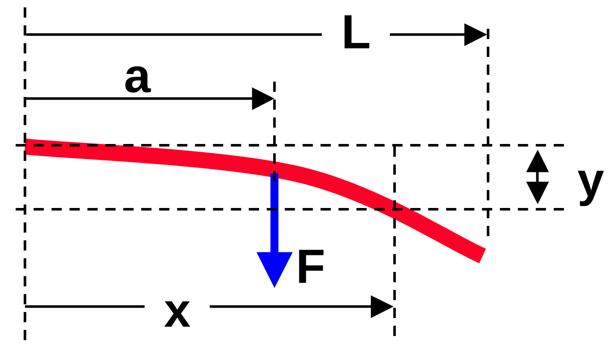
  - **E is Young's modulus**
  - **I is bending moment of inertia**
    - **for a rectangular cross section I is**

$$I = \frac{1}{12} \cdot w \cdot t^3$$

  - **note maximum displacement is at position L**

$$y_{end}^{max} = \frac{F}{6 \cdot E \cdot I} \cdot a^2 \cdot (3 \cdot L^2 - a)$$

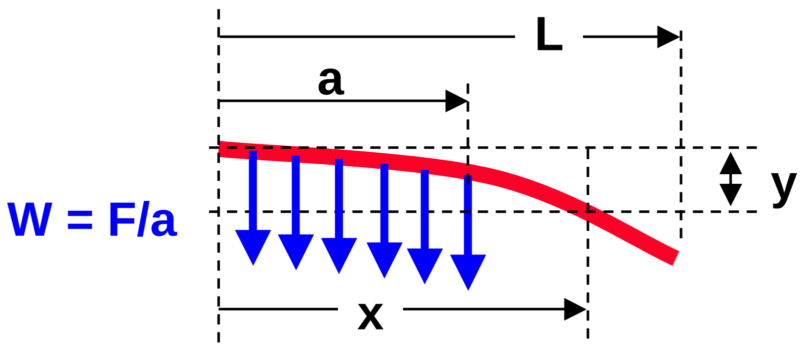
  - **note deflection decreases as cube of thickness**



beam calculator at:
http://www.ecalcx.com/beamanalysis/beamcantpoint_in.asp
other calculators at: http://www.ecalcx.com/

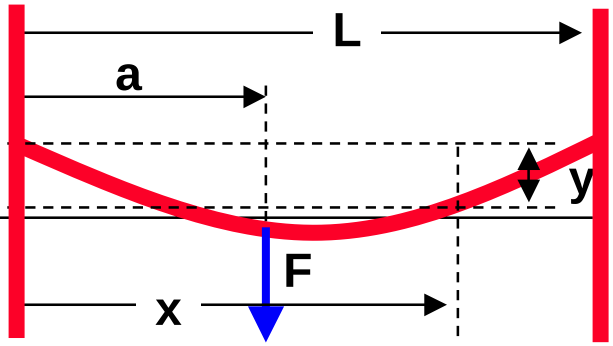
Bending of a simple cantilever beam

- for a uniformly distributed force
  - W = (total force) / a



$$y(x) = \frac{W}{24 \cdot E \cdot I} \cdot \begin{cases} x^2 \cdot \left(6 \cdot a^2 - 4 \cdot a \cdot x + x^2\right) & x < a \\ a^3 \cdot (4 \cdot x - a) & x > a \end{cases}$$

Beam fixed both ends



- **for beam fixed at both ends, point load**

y_x is the vertical deflection of the beam

if $x < a$, $y_x = -(3 M_{x=0} x^2 + R_{x=0} x^3) / (6 E I_z)$

if $x >= a$, $y_x = -(3 M_{x=0} x^2 + R_{x=0} x^3 - W (x - a)^3) / (6 E I_z)$

$M_{x=0}$ is the bending moment reaction at the left hand support
$M_{x=0} = - F a (L - a)^2 / L^2$

$R_{x=0}$ is the vertical reaction force at the left hand end support
$R_{x=0} = F (L - a)^2 (L + 2 a) / L^3$

at load $y_{x=a} = -(3 M_{x=0} a^2 + R_{x=0} a^3) / (6 E I_z)$

y_{max} is the maximum deflection of the beam :
if $a < L / 2$, $y_{max} = 2 F a^2 (L - a)^3 / ((3 L - 2 a)^2 (3 E I_z))$
if $a >= L / 2$, $y_{max} = 2 F a^3 (L - a)^2 / ((L + 2 a)^2 (3 E I_z))$

x_{ymax} is the horizontal location maximum vertical deflection
if $a < L / 2$, $x_{ymax} = L - 2 L (L - a) / (3 L - 2 a)$
if $a >= L / 2$, $x_{ymax} = 2 L a / (L + 2 a)$

Beam fixed both ends

- **for beam fixed at both ends, distributed uniform load to a**

Deflection y_x is the vertical deflection of the beam at x:

if $x < a$, $y_x = -(3 M_{x=0} x^2 + R_{x=0} x^3 - w x^4 / 4) / (6 E I_z)$

if $x >= a$, $y_x = -(3 M_{x=0} x^2 + R_{x=0} x^3 - w x^4 / 4 + w (x - a)^4 / 4) / (6 E I_z)$

$M_{x=0}$ is the bending moment reaction at left support:

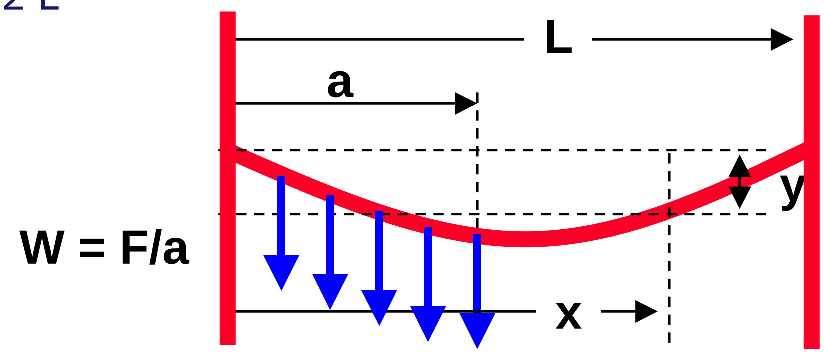
$$M_{x=0} = R_{x=L} L + M_{x=L} - w a^2 / 2$$

$R_{x=0}$ is the vertical reaction force at the left support:

$$R_{x=0} = w a - R_{x=L}$$

$R_{x=L}$ is the vertical reaction force right support:

$$R_{x=L} = w a^3 (2 L - a) / 2 L^3$$



Deflection of a circular diaphragm

- much thinner than radius r
- for pinned around circumference, uniform force per unit area (i.e., uniform pressure P), no built in stress

$$\delta_{center} = \frac{3 \cdot P \cdot r^4 \cdot \left(1 - \nu^2\right)}{16 \cdot E \cdot t^3}$$

Thin film on thick substrate

- **if film is stressed (stress σ), overall curvature results**
  - **E: Young's moduls; ν: Poisson's ratio; t_{sub}: substrate thickness; t_{film}: film thickness; r: radius of curvature**

$$\sigma \approx \frac{E}{1-\nu} \cdot \frac{\left(t_{sub}\right)^2}{t_{film}} \cdot \frac{1}{6 \cdot r}$$

[1] A. Sinha, H. Levinstein, and T. Smith, "Thermal Stresses and Cracking Resistance of Dielectric Films on Si Substrates," *Journal of Applied Physics*, vol. 49, pp. 2423-2426, 1978.
[2] G. Stoney, "The Tension of Metallic Films Deposited by Electrolysis," *Proceedings of the Royal Society*, vol. A82, pp. 172, 1909.

Dynamic response

- **response of generic structure is approximately**

$$\underbrace{m\frac{d^2x}{dt^2}}_{\substack{\text{mass*accel}\\ \text{NSL}}} + \underbrace{b\frac{dx}{dt}}_{\substack{\text{force} \propto \text{velocity}\\ \text{viscous damping}}} + \underbrace{k \cdot x}_{\substack{\text{elastic force}\\ \text{Hooke's law}}} = F_{external}$$

  - **transfer function (Laplace domain)**

$$H(s) = \frac{1/m}{s^2 + \left(\frac{b}{m}\right)\cdot s + \frac{k}{m}}$$

  - **this is the same as an LRC circuit**

$$H(s) = \frac{1/LC}{s^2 + \left(\frac{1}{RC}\right)\cdot s + \frac{1}{LC}} = \frac{\omega_o{}^2}{s^2 + \left(\frac{\omega_o}{Q}\right)\cdot s + \omega_o{}^2}$$