Electrostatic/magnetostatic forces

- **simplest approach: energy method**
  - **recall that energy = force (vector) • "travel" (vector)**
  - **then**

$$F = \frac{\partial(\text{energy})}{\partial(\text{distance})}$$

  - **note that this can give the TOTAL force (not pressure) if you can identify a single spatial coordinate that is parallel to the force**
  - **simple example: parallel plates**
    - **electrostatic: applied voltage V**
    - **magnetostatic: current I**

Parallel plate capacitors

- here the fields are uniform, fringe fields are "small"
  - force equation is found using

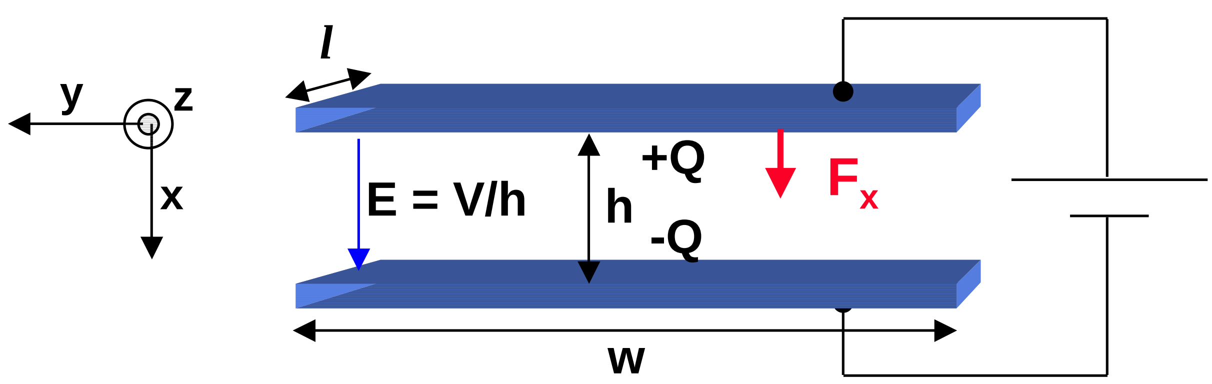
$$U_{cap} = \frac{1}{2} C \cdot V^2 = \frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{h}\right) \cdot V^2$$

$$\boxed{\left|F_x\right| = \frac{\partial U_{cap}}{\partial x} = \frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{h^2}\right) \cdot V^2}$$

  - or more simply

$$\left|F_x\right| = Q_{plate} \cdot \left|E\right| = \left(\frac{C \cdot V}{2}\right) \cdot \frac{V}{h} = \left(\frac{1}{2}\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{h} \cdot V\right) \cdot \frac{V}{h} = \frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{h^2}\right) \cdot V^2$$

  - force is attractive between plates



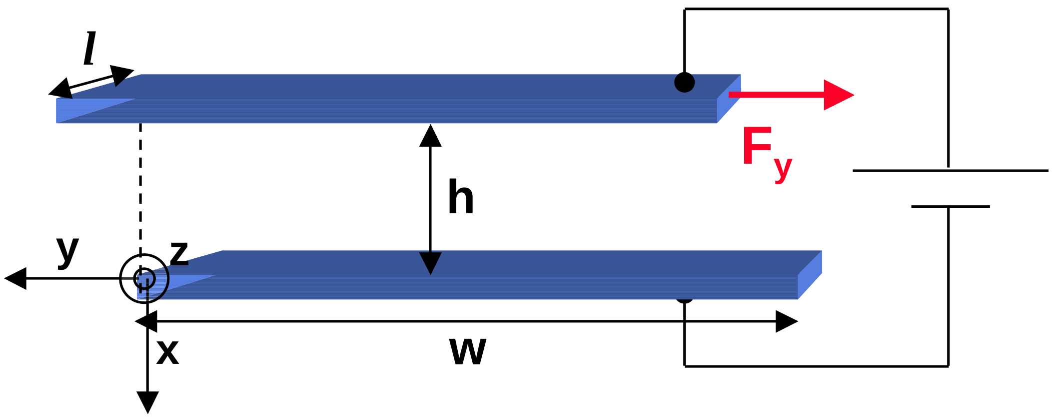
Lateral displacement

- **consider case where one plate is displaced a distance y wrt other plate**
  - **ignoring fringe fields**

$$C(y) = \frac{\varepsilon_r \cdot \varepsilon_o \cdot (w - y) \cdot l}{h}$$

$$U_{cap} = \frac{1}{2} C \cdot V^2 = \frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot (w - y) \cdot l}{h}\right) \cdot V^2$$

$$F_y = \frac{\partial U_{cap}}{\partial y} = \frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot l}{h}\right) \cdot V^2 \cdot \begin{cases} -1 & y > 0 \\ +1 & y < 0 \end{cases}$$

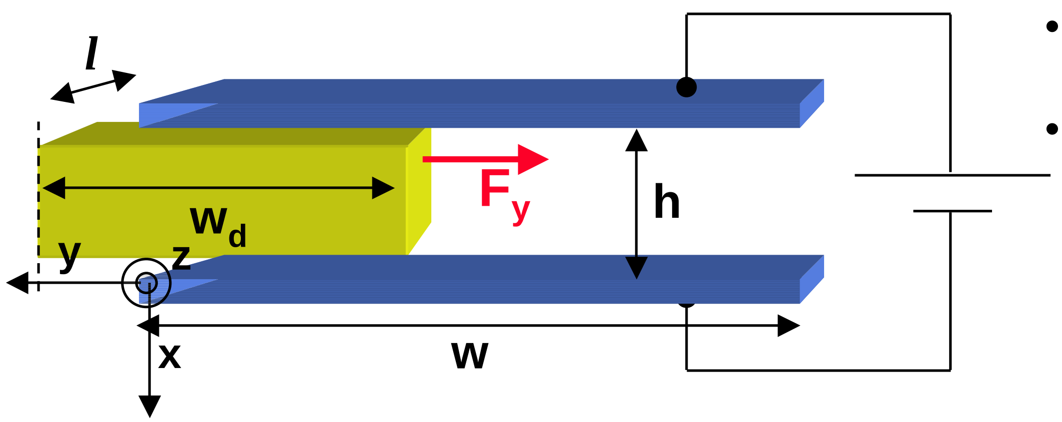


- **force is indep of displacement**
- **reverses sign at y = 0**
  - **restoring force centers the plates**

Lateral displacement of dielectric slab

- **consider case where dielectric slab (of thickness h) is displaced a distance y wrt the plates**
  - **ignoring fringe fields** $C(y) = \frac{\varepsilon_r \cdot \varepsilon_o \cdot (w_d - y) \cdot l}{h} + \frac{\varepsilon_o \cdot [w - (w_d - y)] \cdot l}{h}$

$$F_y = \frac{\partial U_{cap}}{\partial y} = \frac{1}{2}\left[-\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot l}{h}\right) + \left(\frac{\varepsilon_o \cdot l}{h}\right)\right] \cdot V^2 = -\frac{1}{2}\frac{\varepsilon_o \cdot l}{h}[\varepsilon_r - 1] \cdot V^2 \quad (y > 0)$$

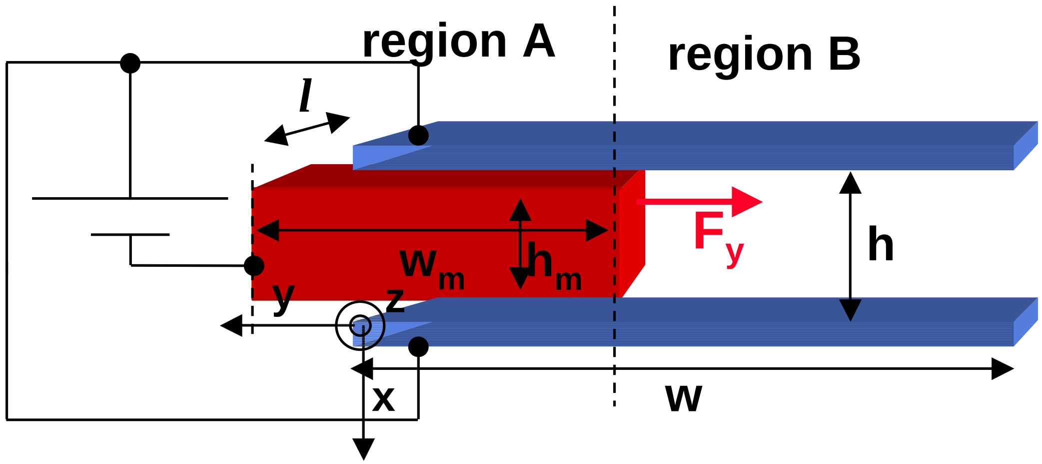


- **force is indep of displacement**
- **is zero at y = 0**
  - **force pulls slab into region between the plates**

Lateral displacement of metallic slab

- consider case where BIASED metallic slab is displaced a distance y wrt the plates
  - ignoring fringe fields

$$C = C_A = \frac{\varepsilon_r \cdot \varepsilon_o \cdot (w_m - y) \cdot l}{h - h_m}$$



$$F_y = \frac{\partial U_{cap}}{\partial y} = \frac{1}{2}\left[-\frac{\varepsilon_r \cdot \varepsilon_o \cdot l}{h - h_m}\right] \cdot V^2 \qquad (y > 0)$$

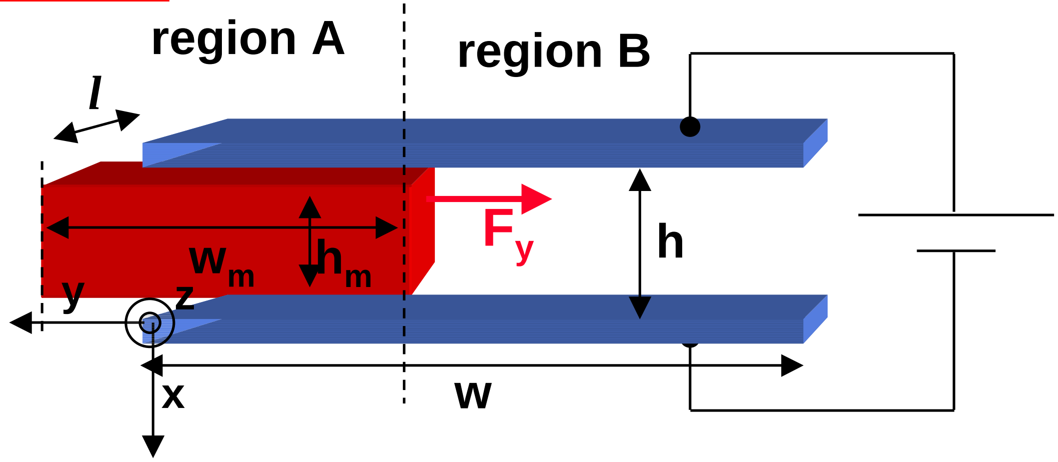
- note there is NO net force in the x direction!

Lateral displacement of metallic slab

- consider case where floating metallic slab is displaced a distance y wrt the plates
  - ignoring fringe fields

$$C_A = \frac{\varepsilon_r \cdot \varepsilon_o \cdot (w_m - y) \cdot l}{h - h_m}$$

$$C_B = \frac{\varepsilon_r \cdot \varepsilon_o \cdot [w - (w_m - y)] \cdot l}{h}$$

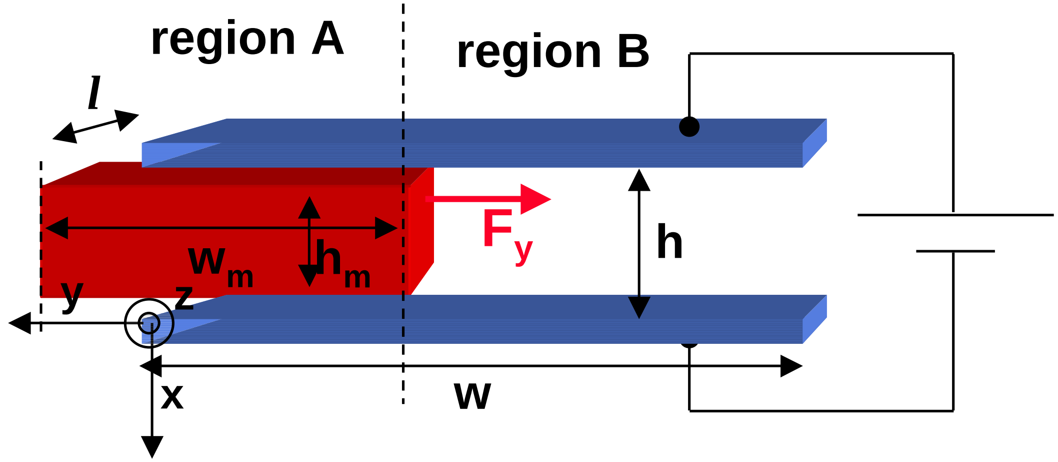


$$C_{total} = \left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot (w_m - y) \cdot l}{h - h_m}\right) + \left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot [w - (w_m - y)] \cdot l}{h}\right)$$

Lateral displacement of metallic slab

- **consider case where metallic slab is displaced a distance y wrt the plates**
  - **ignoring fringe fields**

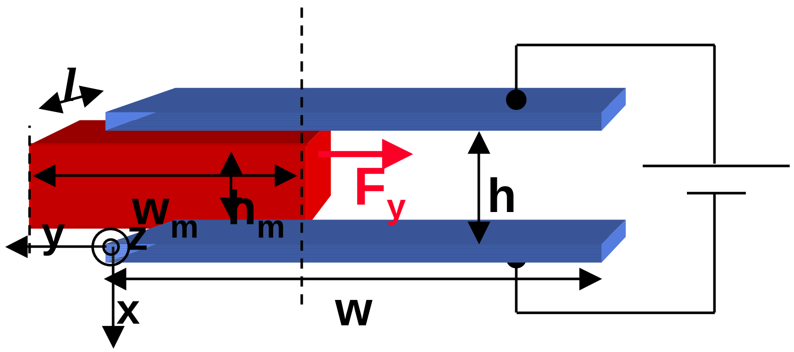
$$C_{total} = \varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left(\frac{(w_m - y)}{h - h_m} + \frac{[w - (w_m - y)]}{h} \right)$$

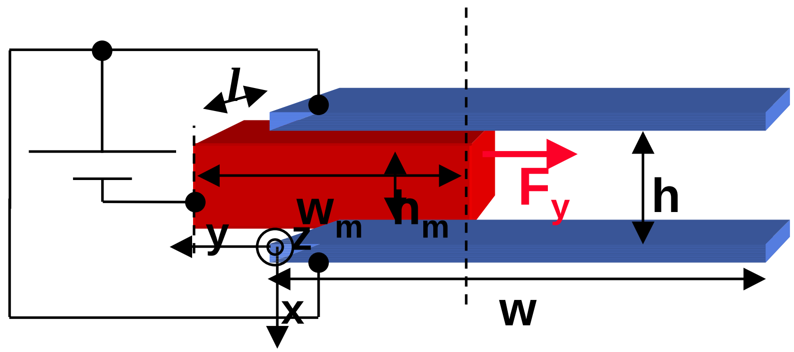


$$\frac{\partial C_{total}}{\partial y} = \varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left[\frac{-1}{h - h_m} + \frac{1}{h} \right] = \varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left[\frac{-h_m}{h \cdot (h - h_m)} \right]$$

$$F_y = \frac{\partial U_{cap}}{\partial y} = \frac{1}{2} \frac{\partial C}{\partial y} \cdot V^2 = -\frac{1}{2} \varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left[\frac{h_m}{h \cdot (h - h_m)} \right] \cdot V^2 \quad (y > 0)$$

Comparison of two bias schemes





$$F_y = -\frac{1}{2}\varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left[\frac{h_m/h}{(h-h_m)}\right] \cdot V^2$$

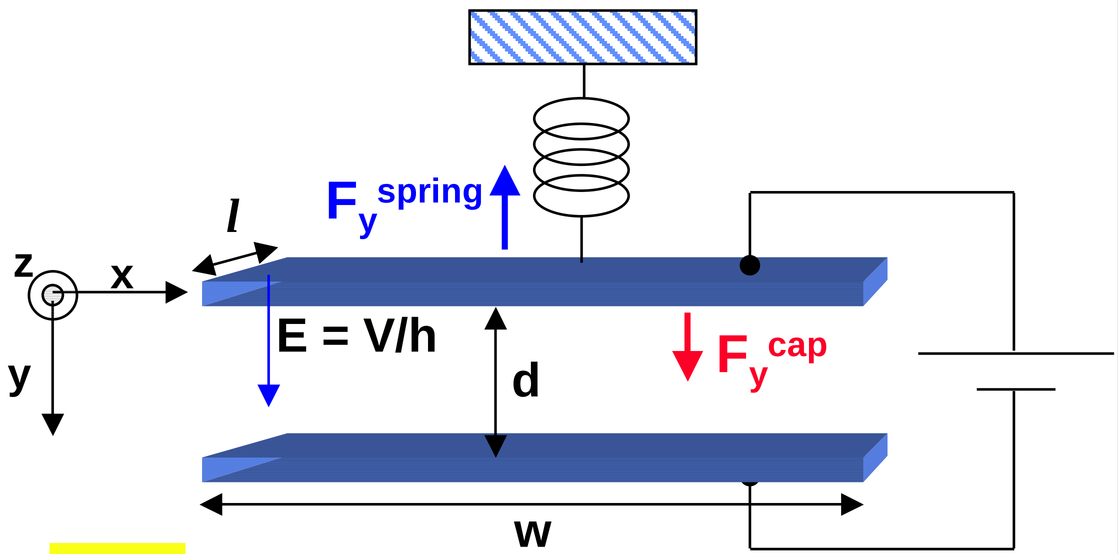
$$F_y = -\frac{1}{2}\varepsilon_r \cdot \varepsilon_o \cdot l \cdot \left[\frac{1}{h-h_m}\right] \cdot V^2$$

- **"unbiased" metal slab force < "biased" metal slab**
- **no force in x direction (in absence of fringe fields)**
- **force is zero once slab is fully "between" the plates**

Electrostatic – mechanical force balance

- assume one plate fixed, other connected to a spring
  - force balance

$$\underbrace{\frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{(d-y)^2}\right)\cdot V^2}_{\left|F_x^{\text{cap}}\right|} = \underbrace{k \cdot y}_{\left|F_x^{\text{spring}}\right|}$$



$$y = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2}{2 \cdot k} \cdot \frac{L}{(d-y)^2}$$

use normalized displacement $\Delta \equiv \frac{y}{d}$

$$\Delta = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2}{2 \cdot k} \cdot \frac{L}{d^2} \cdot \frac{1}{(1-\Delta)^2}$$

$$\Delta \cdot (1-\Delta)^2 = \underbrace{\frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2}{2 \cdot k} \cdot \frac{L}{d^2}}_{\text{normalized force}}$$

cubic, solve implicitly

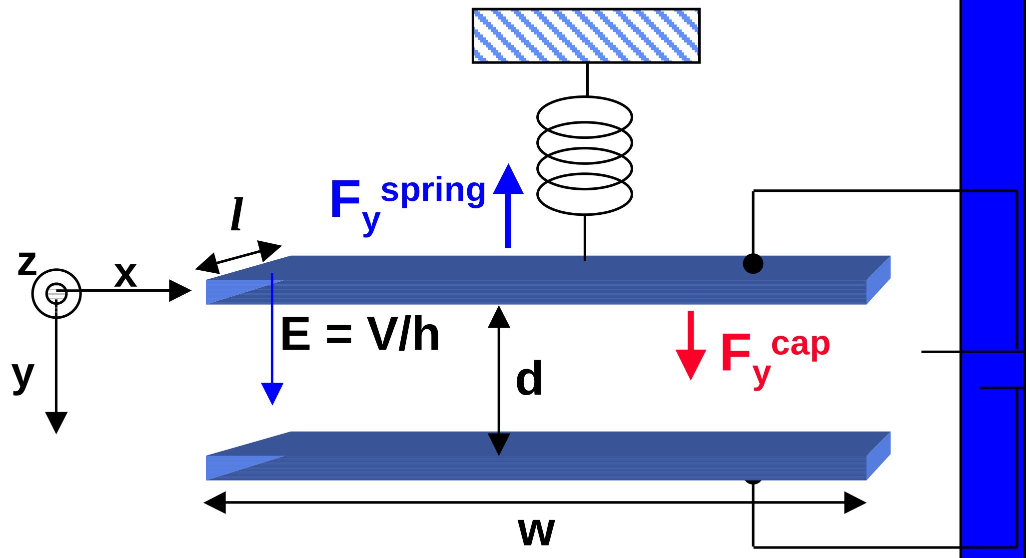
Electrostatic – mechanical force balance

- obvious problem
  - max force from spring is k•d, but $F_{electro} \Rightarrow \infty$ as $y \Rightarrow d$
  - estimate bound on y:

$$\underbrace{\frac{1}{2}\left(\frac{\varepsilon_r \cdot \varepsilon_o \cdot A}{(d - y_{max})^2}\right)\cdot V^2}_{\left|F_x^{cap}\right|} < \underbrace{k \cdot d}_{\left|F_{max}^{spring}\right|}$$

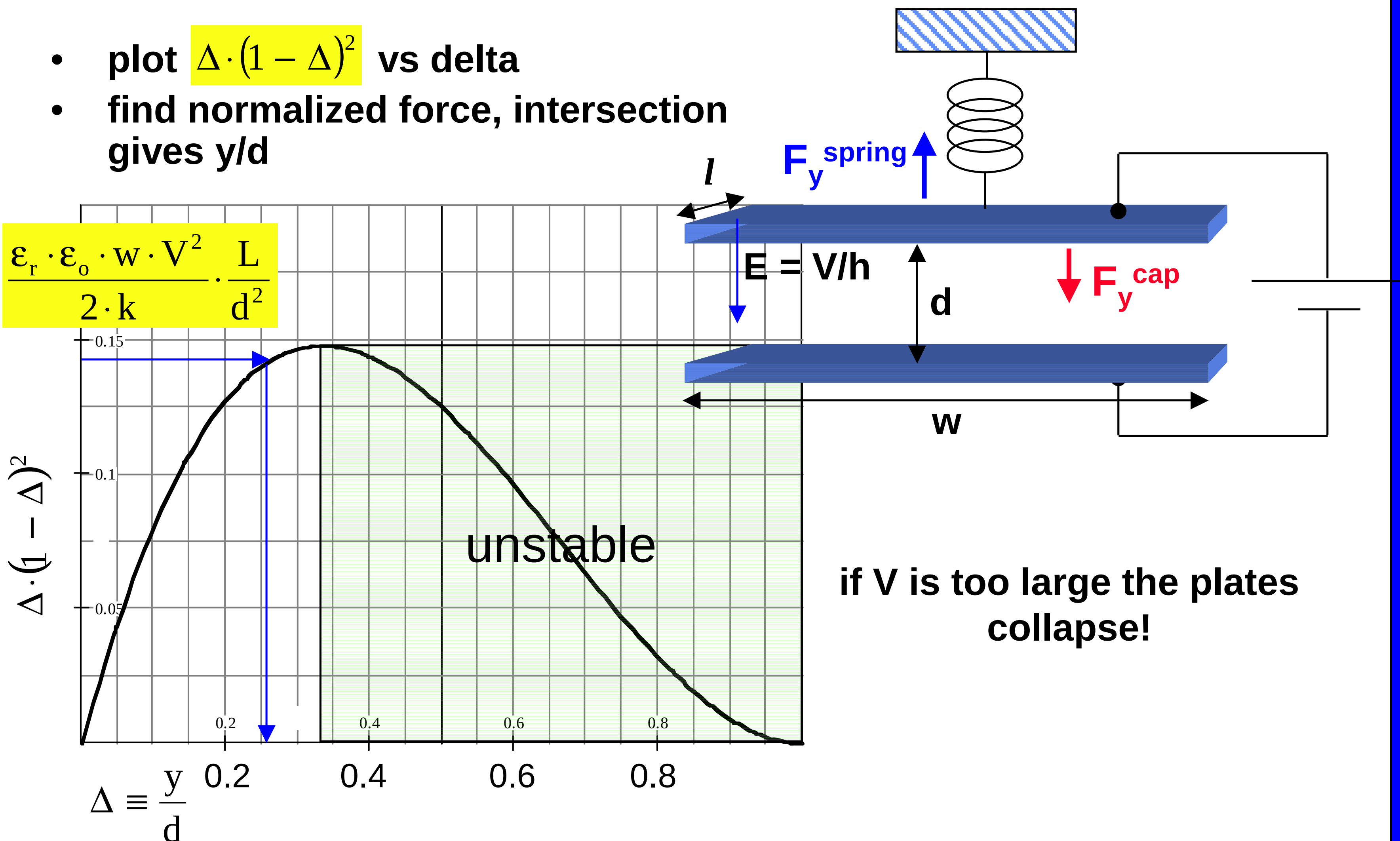
$$(d - y_{max})^2 > \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot L \cdot V^2}{2 \cdot k \cdot d}$$

$$\frac{y_{max}}{d} < \left(1 - \sqrt{\frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot L \cdot V^2}{2 \cdot k \cdot d^3}}\right)$$



Electrostatic – mechanical force balance

- plot $\Delta \cdot (1-\Delta)^2$ vs delta
- find normalized force, intersection gives y/d



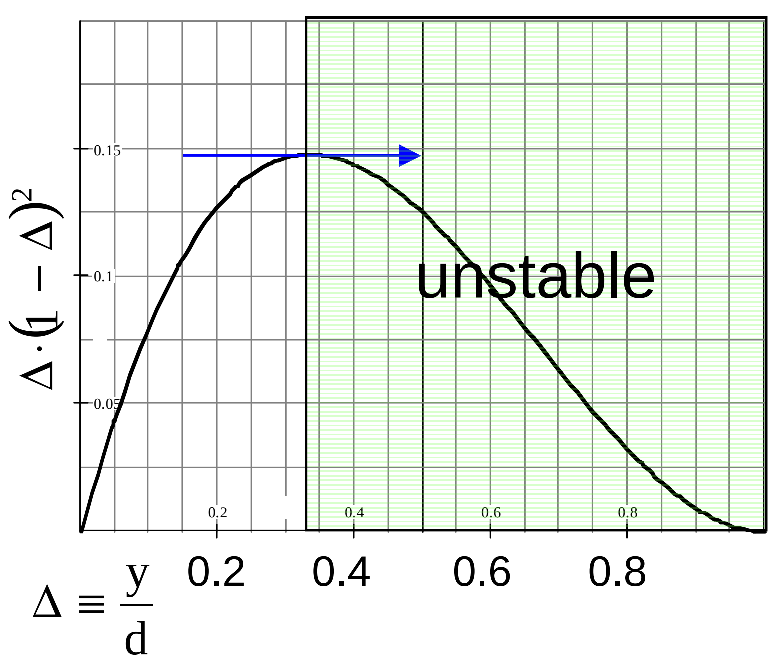
if V is too large the plates collapse!

Electrostatic – mechanical force balance

- **if applied voltage exceeds some maximum, the attractive force exceeds the restoring force from the spring**
  - **max at**

$$\frac{\partial}{\partial\Delta}\Delta\cdot(1-\Delta)^2 = 0 \Rightarrow (1-\Delta)^2 - 2\cdot\Delta\cdot(1-\Delta) = 0 \Rightarrow (1-\Delta)\cdot(1-3\Delta) = 0$$

$$\Rightarrow \Delta_{max} = \frac{1}{3}$$



max displacement is 1/3 of initial separation

$$V_{max} = \sqrt{\frac{2\cdot k}{\varepsilon_r\cdot\varepsilon_o\cdot w}\cdot\frac{d^2}{L}\cdot\Delta_{max}\cdot(1-\Delta_{max})^2}$$

$$\Rightarrow V_{max} = \frac{2}{3}\sqrt{\frac{2}{3}}\sqrt{\frac{k}{\varepsilon_r\cdot\varepsilon_o\cdot w\cdot L}}\cdot d$$

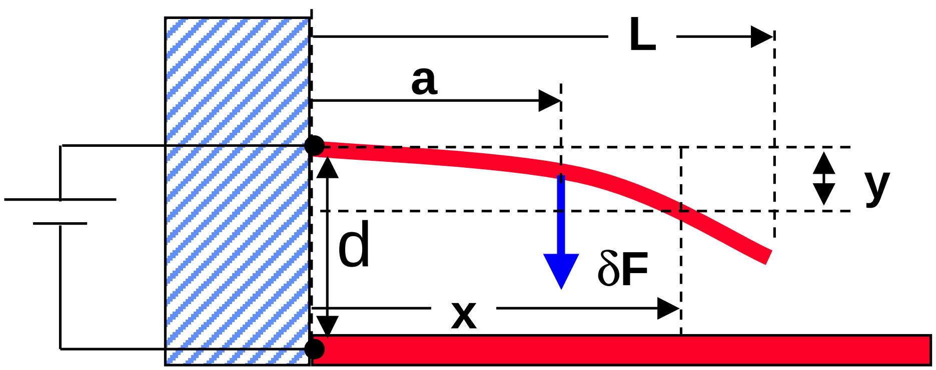
Electrostatic displacement of cantilever

- **capacitor formed by cantilever beam and fixed plate**
- **cantilever beam: supported at one end only**
  - **point force F at position a, displacement y at position x**

$$y(x) = \frac{F}{6 \cdot E \cdot I} \cdot \begin{cases} x^2 \cdot (3 \cdot a - x) & x < a \\ a^2 \cdot (3 \cdot x - a) & x > a \end{cases} \qquad I = \frac{1}{12} \cdot w \cdot t^3$$

  - **tip deflection (x = L) due to δF at position x = a**

$$\delta y_{tip} = \frac{\delta F}{6 \cdot E \cdot I} \cdot a^2 \cdot (3 \cdot L - a)$$



Electrostatic beam displacement

- **want to integrate to get total tip deflection**

$$y_{tip} = \int_0^L \left\{ \frac{dF}{6 \cdot E \cdot I} \cdot x^2 \cdot (3 \cdot L - x) \right\}$$

- **recall force per area for separation h is** $P = \dfrac{\varepsilon_r \cdot \varepsilon_o}{2} \left(\dfrac{V}{h} \right)^2$

- **force δF due to electrostatic attraction at x**

$$dF = \frac{\varepsilon_r \cdot \varepsilon_o}{2} \left(\frac{V}{d - y(x)} \right)^2 \cdot \underbrace{w \cdot dx}_{\text{area}}$$

- **but from beam equation the displacement y at the point of force application x**

$$y(x) = \frac{dF}{3 \cdot E \cdot I} \cdot x^3$$

- **so solve for dF as function of x**
  - **requires solution of a cubic!**

$$dF = \frac{\varepsilon_r \cdot \varepsilon_o}{2} \left(\frac{V}{d - \dfrac{dF}{3 \cdot E \cdot I} \cdot x^3} \right)^2 \cdot w \cdot dx$$

Electrostatic beam displacement

- want to integrate to get total tip deflection

$$y_{tip} = \int_0^L \left\{ \frac{dF}{6 \cdot E \cdot I} \cdot x^2 \cdot (3 \cdot L - x) \right\}$$

- try something simpler
  - assume "parabolic" bending of beam

$$y(x) = \left(\frac{x}{L} \right)^2 \cdot y_{tip}$$

- then force δF due to electrostatic attraction at x

$$dF = \frac{\varepsilon_r \cdot \varepsilon_o}{2} \left(\frac{V}{d - (x/L)^2 \cdot y_{tip}} \right)^2 \cdot w \cdot dx$$

- so the total deflection of the tip is

$$y_{tip} = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2}{2 \cdot 6 \cdot E \cdot I} \int_0^L \left\{ \frac{3 \cdot L \cdot x^2 - x^3}{\left[d - \left(y_{tip} / L^2 \right) \cdot x^2 \right]^2} \right\} dx$$

- solve "implicitly" by assume a value for y_{tip} inside the integral, then integrate, compare integral to assumed value

Electrostatic beam displacement

- "parameterize" problem using y_{tip} = Y inside integral
- using parabolic approx: $$y_{tip} = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2}{2 \cdot 6 \cdot E \cdot I} \int_0^L \left\{ \frac{3 \cdot L \cdot x^2 - x^3}{\left[d - \left(Y/L^2 \right) \cdot x^2 \right]^2} \right\} dx$$
- integrals:
  - $$\int \left\{ \frac{x^2}{\left[a^2 - x^2 \right]^2} \right\} dx = \frac{x}{2\left(a^2 - x^2\right)} - \frac{1}{4a} \cdot \ln\left| \frac{a+x}{a-x} \right|$$
  - $$\int \left\{ \frac{x^3}{\left[a^2 - x^2 \right]^2} \right\} dx = \frac{a^2}{2\left(a^2 - x^2\right)} + \frac{1}{2} \cdot \ln\left| a^2 - x^2 \right|$$

Electrostatic beam displacement

- **after integration replace Y with y_{tip}**

$$y_{tip} = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2 \cdot L^4}{2 \cdot 6 \cdot E \cdot I \cdot \left(y_{tip}\right)^2}\left[\frac{3L^2 - \left(d \cdot L^2 / y_{tip}\right)}{2\left\{\left(d \cdot L^2 / y_{tip}\right) - L^2\right\}} + \frac{1}{2} - \frac{3L}{4\sqrt{d \cdot L^2 / y_{tip}}} \cdot \ln\left|\frac{\sqrt{d \cdot L^2 / y_{tip}} + L}{\sqrt{d \cdot L^2 / y_{tip}} - L}\right| + \frac{1}{2} \cdot \ln\left|\frac{d \cdot L^2 / y_{tip}}{d \cdot L^2 / y_{tip} - L^2}\right|\right]$$

- **using normalized displacement** $\Delta = \dfrac{y_{tip}}{d}$

- **integral becomes**

$$\Delta^3 = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2 \cdot L^4}{12 \cdot E \cdot I \cdot d^3}\left[\frac{\Delta}{1 - \Delta} + \frac{1}{2} \cdot \ln\left|\frac{1}{1 - \Delta}\right| - \frac{3}{4} \cdot \sqrt{\Delta} \cdot \ln\left|\frac{1 + \sqrt{\Delta}}{1 - \sqrt{\Delta}}\right|\right]$$

- **or finally** $$\Delta^3 \cdot \left[\frac{\Delta}{1 - \Delta} + \frac{1}{2} \cdot \ln\left|\frac{1}{1 - \Delta}\right| - \frac{3}{4} \cdot \sqrt{\Delta} \cdot \ln\left|\frac{1 + \sqrt{\Delta}}{1 - \sqrt{\Delta}}\right|\right]^{-1} = \frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2 \cdot L^4}{12 \cdot E \cdot I \cdot d^3}$$

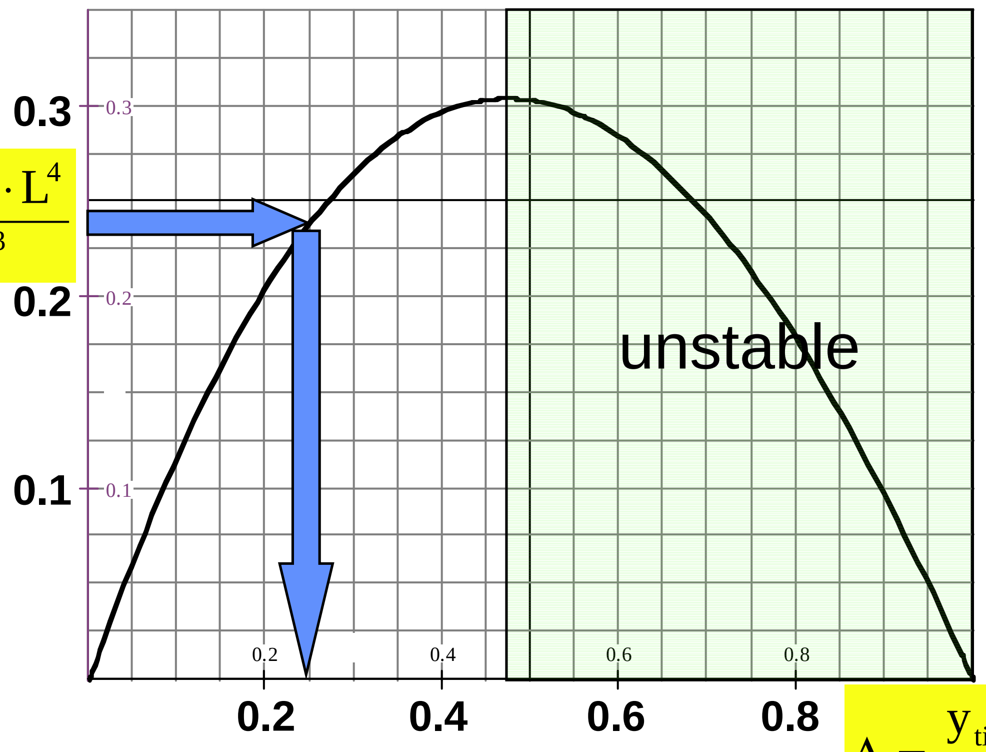
Electrostatic beam displacement

- **solve implicitly by plotting vs delta**

$$\Delta^3 \cdot \left[\frac{\Delta}{1-\Delta} + \frac{1}{2} \cdot \ln\left|\frac{1}{1-\Delta}\right| - \frac{3}{4} \cdot \sqrt{\Delta} \cdot \ln\left|\frac{1+\sqrt{\Delta}}{1-\sqrt{\Delta}}\right| \right]^{-1}$$

- **solution is obtained by plotting "normalized force"**
  - **then find value of delta at intersection**

$$\frac{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot V^2 \cdot L^4}{12 \cdot E \cdot I \cdot d^3}$$



$$\Delta_{max} \approx 0.472$$

$$V_{max} \approx \sqrt{0.472 \cdot \frac{12 \cdot E \cdot I \cdot d^3}{\varepsilon_r \cdot \varepsilon_o \cdot w \cdot L^4}}$$

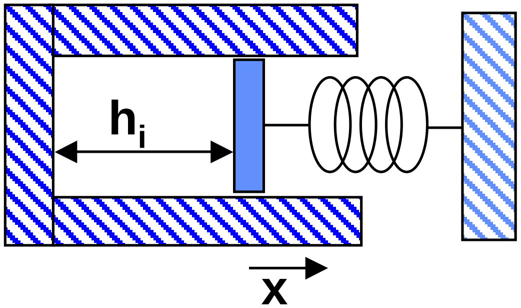
$$\Delta = \frac{y_{tip}}{d}$$

Thermopneumatic actuation

- **use sealed volume**
  - **temperature change accompanied by pressure and volume changes**

$$P_i \cdot V_i = N \cdot R \cdot T_i$$

$$V_i = A \cdot h_i$$



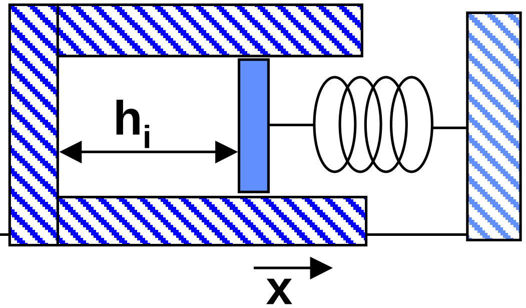
$$P_f \cdot V_f = N \cdot R \cdot T_f$$

$$V_f = A \cdot (h_i + x)$$

$$\frac{P_i \cdot V_i}{T_i} = \frac{P_f \cdot V_f}{T_f} \Rightarrow \frac{P_i \cdot A \cdot h_i}{T_i} = \frac{P_f \cdot A \cdot (h_i + x)}{T_f} \Rightarrow \frac{h_i + x}{h_i} = \frac{P_i}{P_f} \cdot \frac{T_f}{T_i}$$

but $$P_f = P_i + \frac{k \cdot x}{A} \Rightarrow \frac{h_i + x}{h_i} = \frac{P_i}{P_i + \frac{k}{A} \cdot x} \cdot \frac{T_f}{T_i}$$

Thermopneumatic actuation



- **resulting equation for displacement is quadratic**

$$\frac{h_i + x}{h_i} = \frac{P_i}{P_i + \frac{k}{A}\cdot x}\cdot\frac{T_f}{T_i} \quad\Rightarrow\quad x^2\cdot\frac{k}{A} + x\cdot\left(\frac{k\cdot h_i}{A}+P_i\right) + h_i\cdot P_i\cdot\left(1-\frac{T_f}{T_i}\right) = 0$$

  - **solve for x, using the normalized "force"** α

$$\alpha \equiv \frac{P_i\cdot A}{k\cdot h_i}$$

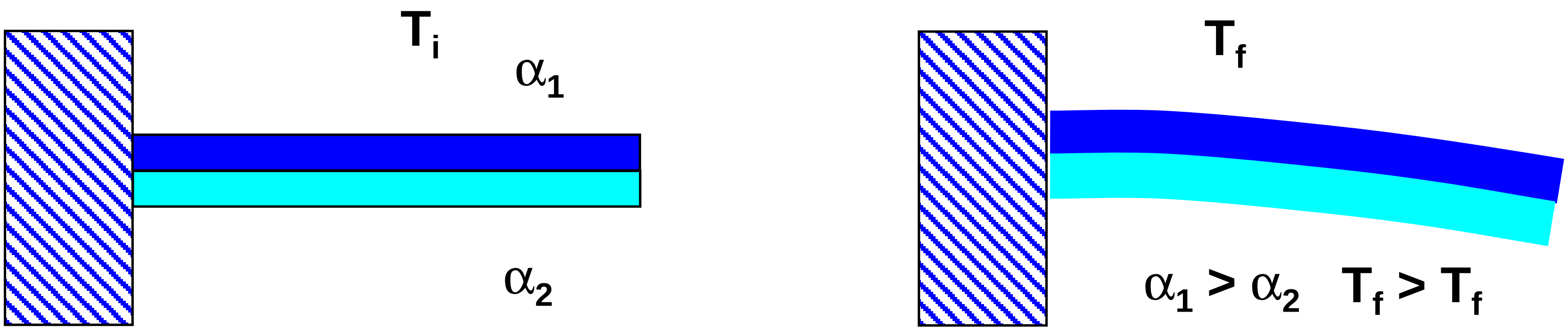
$$x = \frac{1}{2}\cdot h_i\cdot(1+\alpha)\cdot\left[-1+\sqrt{1+4\cdot\frac{\alpha}{1+\alpha}\cdot\left(\frac{T_f}{T_i}-1\right)}\right]$$

- **to maximize displacement sensitivity to temperature make** α **big**
  - **(cross sectional area /** h_i**) big**
  - **k small**
- **leading term**

$$h_i\cdot(1+\alpha) = h_i + \frac{P_i\cdot A}{k}$$

Thermal deflection of bimorph beam

- **consider a cantilever beam made of two materials with un-equal TCE's**
  - **assume no built-in stress, one end pinned cantilever**



$$R \approx \frac{(t_1 + t_2)^2}{6\cdot(\alpha_1 - \alpha_2)\cdot(T_f - T_i)\cdot t_1 \cdot t_2}$$

Kovacs (units are wrong)

$$R \approx \frac{4\cdot(t_1 + t_2)^2 - 2t_1\cdot t_2 + \dfrac{E_1\cdot w_1\cdot(t_1)^3}{E_2\cdot w_2\cdot t_2} + \dfrac{E_2\cdot w_2\cdot(t_2)^3}{E_1\cdot w_1\cdot t_1}}{6\cdot(\alpha_1 - \alpha_2)\cdot(T_f - T_i)\cdot(t_1 + t_2)}$$

Tabib-Azar